

Financial Integration, Liquidity and the Depth of Systemic Crises*

Fabio Castiglionesi[†] Fabio Feriozzi[‡] Guido Lorenzoni[§]

November 2008

Abstract

We investigate how financial integration affects banks' liquidity holding and the depth of systemic crises. We model a multiple regions economy, where each region is characterized by idiosyncratic liquidity shocks. The liquidity shocks are not necessarily asymmetric, then liquidity coinsurance is not always possible. The first-best allocation is reached when the regions are completely integrated since in this case the perfect liquidity coinsurance is achieved. Under complete integration no crisis would occur. However, when financial integration is only partial the perfect liquidity coinsurance is no longer possible. Nevertheless, two (or more) regions will find it optimal to coinsure the liquidity shocks in order to reduce their liquidity uncertainty and to increase their expected utility. Accordingly, under partial integration, the regions could find it optimal to reduce the liquidity holding and to increase the level of long term investment. In this case, partial integration opens up the opportunity for the occurrence of extreme events. That is, the cost of liquidity can become unusually high and the optimal consumption can display both higher volatility than in autarky and negative skewness. When complete financial integration is not achievable, then extreme events can be the optimal outcome of partial integration.

JEL Classification: G21.

Keywords: Financial Integration, Systemic Crises, Liquidity Coinsurance, Skewness.

*We thank Wolf Wagner and seminar participants at LUISS (Rome), Bundesbank (Frankfurt) and Instituto de Empresa (Madrid) for useful comments. The usual disclaimer applies.

[†]CentER, Department of Finance, Tilburg University. E-mail: fabio.castiglionesi@uvt.nl.

[‡]CentER, Department of Finance, Tilburg University. E-mail: f.feriozzi@uvt.nl.

[§]MIT, Department of Economics. E-mail: glorenzo@mit.edu.

1 Introduction

It is usually claimed that globalization, or the process of increasing financial integration, exposes the financial system to the occurrence of “black swan” or extreme events, such as deep systemic crises. This paper shows that this claim is true but it could be the optimal response of agents that live in a world where perfect financial integration is not achieved.

The main idea is the following. Consider a four-region economy where, in each region, the banking sector has access to a liquid short-term asset and a safe illiquid long-term asset and consumers give their endowment to the banks in order to exploit such investment opportunities. The four regions can have negatively correlated liquidity needs so that there are gains from trade from pooling the financial resources, for example through an interbank deposit market.

However, the four regions can be characterized by different degrees of financial integration. We first consider a perfectly integrated financial system, that is the four regions have access to the interbank deposit market. In such environment, systemic crises would never occur since there would always be the possibility to coinsure against idiosyncratic shocks and, consequently, there would not be any residual aggregate uncertainty.

We then consider a completely disconnected financial system. That is, none of the regions have access to the interbank deposit market. In autarky, each region faces aggregate uncertainty and it is exposed to the highest variability of liquidity shocks. Clearly, under this scenario, consumers get a lower expected utility than under the complete integration case. We then analyze a situation of partial integration. That is, only two regions can trade in the interbank deposit market. In this case, since liquidity shocks can be negatively correlated, the two regions find it useful to pool financial resources getting liquidity coinsurance. However, idiosyncratic liquidity shocks cannot be insured when they are positively correlated. Consequently, residual aggregate uncertainty is reduced with respect the autarky but cannot be completely eliminated.

Under partial integration, consumer get an higher expected utility than the one achieved under autarky. The reason being that the volatility of liquidity shocks faced in partial integration is lower than the same volatility in autarky. This is a direct implication of the reduced residual aggregate uncertainty under partial integration. As a consequence, consumers prefer partial integration to autarky. So, whenever possible, regions will take advantage of the risk-sharing opportunities offered by the interbank deposit market.

The welfare improving opportunity offered by partial integration has direct implication on the optimal investment decision of the banks. It turns out that the optimal investment in the liquid short-term asset can be decreasing in the amount of residual aggregate uncertainty. That is, when the regions have the possibility to partially integrate they will reduce the investment in liquidity compared with the autarky situation. The reason is that less residual aggregate uncertainty implies less need of liquidity, which under partial

integration is provided also by the interbank market. The regions then find it optimal to increase the investment in the long-term asset in order to increase consumers' expected utility.

The optimal investment decision has direct consequences on the occurrence of extreme events. Indeed, under partial integration, aggregate uncertainty is not completely eliminated. The systemic crisis can then occur when both regions face the same (high) liquidity shock. Both financial and real effects are in place. The financial effect can be analyzed by mean of the interest rate on the interbank deposit market, that is the cost of liquidity. It turns out that under partial integration the cost of liquidity is always lower than the complete integration case. The only exception being when the systemic crisis hits. In this case the cost of liquidity spikes up well above the one under complete integration.

The real effect is measured by the distribution of optimal consumption. Under partial integration the optimal consumption variance can be higher than the consumption variance under autarky. In particular, this is always true whenever the residual aggregate uncertainty is sufficiently high. Moreover, partial integration opens up the opportunity to negative consumption skewness contrary to autarky and complete integration. That is, it is optimal to observe particularly low level of consumption during systemic crisis.

The analysis of interbank insurance has received widespread attention in the literature. The focus of analysis has been mainly on asymmetric information or moral hazard models. In the paper by Rochet and Tirole [14] interbank markets act as a device that banks use to monitor each other. Bhattacharya and Gale [6] show that banks under-invest in liquidity reserves when moral hazard and adverse selection problems are present. In Freixas et al. [11] interbank markets operate in a spatial economy and a solvency shock can cause a gridlock in the system. Freixas and Holthausen [10] study how asymmetric information affects interbank market integration. Brusco and Castiglionesi [7] analyze the optimal form of interbank insurance considering the risky behavior of banks affected by moral hazard and protected by limited liability. Contrary to this tradition, we characterize the interbank market with an environment where participants have symmetric information.

Allen and Gale [3] show that interbank markets provide optimal liquidity insurance when banks are subject to idiosyncratic shocks, but may expose the system to financial meltdown when an unexpected aggregate liquidity shock hits. Allen and Gale [4] analyzes the optimal liquidity provision by banks and markets under complete and incomplete markets and/or contracts. We analyzed aggregate uncertainty being fully anticipated by banks. The incompleteness we analyze in this paper is the partial integration, that is the situation in which not all the banks can participate in the risk sharing agreement.

Our paper also relates to the recent literature on liquidity provision by interbank markets. These papers highlight the inefficiencies of interbank market in order to explain the lack of liquidity during systemic crises. In Wagner [15] interbank lending may break down due to a moral hazard problem at banks. In Acharya, Gromb and Yorulmazer [1] ineffi-

ciencies in interbank lending arise due to monopoly power. In Allen, Carletti and Gale [2] inefficiencies in the interbank market arise because interest rates fluctuate too much in response to shocks, which precludes efficient risk sharing. In Castiglionesi and Wagner [8] the inefficient liquidity provision is due to non exclusive contract. This paper shows that banks could optimally hold a low amount of liquidity if they are constrained to operate in a partially integrated economy.

Finally, this paper can be considered complementary to the literature on rare events, pioneered by Rietz [13], and recently revived by Barro [5]. The aim of this literature is to explain well known financial puzzles, like the equity premium, taking into account that rare disasters, either real or financial, have small probability to occur. The objective of these papers is mainly empirical, that is to calibrate disaster probabilities, showing that they can generate the evidence we observe. Beside the issue about their robustness in explaining not only financial puzzles but also other financial implications (see Gourio [12] for a more sceptical evidence), these papers take as given that rare events occur and there is no mechanism that explains the depth of the crisis. While focusing our attention on financial crises, our paper establishes a link between partial financial integration, in the form of residual aggregate uncertainty, and the depth of the systemic crisis (when it occurs).

The remainder of the paper is organized as follows. Section 2 presents the four regions economy. Section 3 characterizes the optimal risk-sharing, showing that when the regions are completely integrated the efficient allocation is obtained. Section 4 analyzes the solution both when the regions are fully segmented (i.e., in autarky) and when they are only partially integrated. Section 5 analyzes the consequences of partial integration on the depth of the systemic crisis, both in term of the cost of liquidity in the interbank market (section 5.1) and in term of the variance and skewness of the optimal consumption profile (section 5.2). Section 6 concludes, and the Appendix contains the proofs.

2 The Model

There are three dates ($t = 0, 1, 2$), and a single good that serves as numeraire. There are two types of assets. A liquid asset (the *short asset*) that takes one unit of the good at date t and converts it into one unit of the good at date $t + 1$. An illiquid asset (the *long asset*) that takes one unit of the good at date 0 and transforms it into $R > 1$ units of the good at date 2.

There are four regions (labeled A, B, C and D) that are ex-ante identical but differ in the liquidity shock. Each region contains a continuum of ex-ante identical consumers (depositors) with an endowment of one unit of the consumption good at date 0. Agents are assumed to have Diamond-Dybvig [9] preferences, that is,

$$U(c_1, c_2) = \begin{cases} u(c_1) & \text{with probability } \omega^i; \\ u(c_2) & \text{with probability } (1 - \omega^i). \end{cases}$$

Table 1: Regional liquidity shocks

	A	B	C	D
S_1	ω_H	ω_H	ω_L	ω_L
S_2	ω_L	ω_H	ω_L	ω_H
S_3	ω_H	ω_L	ω_H	ω_L
S_4	ω_L	ω_L	ω_H	ω_H

The probability ω^i is the random fraction of early consumers in region i , and it can take two possible values ω_H and ω_L , with $\omega_H > \omega_L$. The function $u(\cdot)$ is assumed to be a neoclassical utility function that satisfies Inada's conditions. There are four possible states of the world (S_1, S_2, S_3 and S_4). The probability of S_1 and S_4 is equal to $\frac{1-p}{2}$, while the probability of states S_2 and S_3 is equal to $\frac{p}{2}$. The realization of the liquidity preference shocks is state-dependent, and is given in Table 1.

Notice that ex-ante, each region has a probability equal to 1/2 of having a high liquidity shock. Moreover, there is no aggregate uncertainty since the proportion of early consumers in the economy (i.e., considering the 4 regions) is

$$\gamma \equiv \frac{2\omega_H + 2\omega_L}{4} = \frac{\omega_H + \omega_L}{2}$$

in each state of the world. All uncertainty is resolved in $t = 1$, when the state of nature is revealed and each consumer learns whether she is an early or late consumer. Consumer's type is private information.

Finally, in each region there is a continuum of identical banks. Only banks can invest in the illiquid asset. Then, consumers have to deposit their endowment in the banks of their region in order to take advantage of the long term investment opportunities of the economy.

3 Optimal Risk Sharing

The optimal risk-sharing can be characterized by the solution to the planner problem. The planner overcomes the problem of asymmetric liquidity need of the four regions through the direct investment of the resources on the assets. Assume that the planner cannot observe the agent's type. Let y and x be the per capita amounts invested in the short and long assets, respectively. Since consumers are ex-ante identical, the planner maximizes the unweighted sum of consumer's expected utility. Therefore, the planner problem is [FB]:

$$\max_{(x,y,c_1,c_2)} \gamma u(c_1) + (1 - \gamma)u(c_2)$$

subject to the feasibility constraints:

$$x + y \leq 1; \quad \gamma c_1 \leq y; \quad (1 - \gamma)c_2 \leq Rx;$$

$$x \geq 0; \quad y \geq 0; \quad c_1 \geq 0; \quad c_2 \geq 0.$$

It is obvious that optimality requires that the feasibility constraints are satisfied with equality, so we can write the problem as

$$\max_{y \in [0,1]} \gamma u \left(\frac{y}{\gamma} \right) + (1 - \gamma) u \left(\frac{1 - y}{1 - \gamma} R \right). \quad (2)$$

Since u is strictly concave and satisfies the Inada conditions, the solution to problem (2) is unique and interior.

The optimal value $y^* \in (0, 1)$ is obtained from the first order condition

$$u' \left(\frac{y^*}{\gamma} \right) = R u' \left(\frac{R(1 - y^*)}{1 - \gamma} \right), \quad (3)$$

and once y^* has been determined by equation (3) we can use the feasibility constraints to determine the other variables, that is

$$c_1^* = \frac{y^*}{\gamma}, \quad c_2^* = \frac{(1 - y^*)}{1 - \gamma} R, \quad x^* = 1 - y^*. \quad (4)$$

Notice that (3) and (4) imply $u'(c_1^*) = R u'(c_2^*)$, which in turn implies $u'(c_1^*) > u'(c_2^*)$ and $c_2^* > c_1^*$. Thus, the first-best allocation automatically satisfies the incentive constraint $c_2 \geq c_1$. That is, late consumers have no incentive to behave as early consumers and the first-best allocation can be achieved even if the planner cannot observe the consumer's type. We will denote the first-best allocation as $\delta^* = (y^*, x^*, c_1^*, c_2^*)$, and with U^* the expected utility achieved under the first-best contract δ^* .

3.1 Complete Integration

We analyze the situation where all the regions are integrated. In this case the first-best allocation can be attained using an interbank market of deposits (see Allen and Gale, [3]). Since the four regions have negatively correlated liquidity needs, banks belonging to the four regions find it useful to exchange deposits between themselves. In particular, banks in the region hit by the high liquidity shock ω_H withdraw their deposits from the bank in the other region at time 1, and at time 2 the funds move in the opposite direction.

The first-best allocation can be attained by a decentralized banking system using interbank deposits as follows:

- Each bank offers the contract $\delta^* = (y^*, x^*, c_1^*, c_2^*)$ to the consumers *and* the banks of the other region;
- Each bank deposits $(\omega_H - \gamma)$ cents in a bank belonging to another region for each dollar deposited by consumers (and receives a deposit of $(\omega_H - \gamma)$ from a bank of the other region).

The interbank deposits are used as liquidity coinsurance instruments against the liquidity shock. With perfect competition in the banking sector, the equilibrium outcome will be that banks offer the contract yielding the first-best allocation, thus maximizing consumers' expected utility.

Proposition 1 *Under complete integration, exchanging an amount $(\omega_H - \gamma)$ of deposits at $t = 0$, the first-best allocation δ^* can be implemented by a decentralized banking system.*

When the four regions are fully integrated the aggregate uncertainty is eliminated and the first-best allocation is implemented. Since the liquidity shocks in the regions are perfectly negatively correlated, the insurance is perfect.

However, when financial integration is only partial, aggregate uncertainty cannot be fully eliminated. In particular, the liquidity shocks can be either negatively or positively correlated. Consequently, even if the interbank deposits market would still play a useful role in smoothing out the idiosyncratic liquidity shocks, full insurance cannot be achieved.

4 Partial Integration

In this section we consider a situation of incomplete integration. The four regions cannot fully coinsure among themselves. We first analyze the most severe form of partial integration, that is autarky. Then, we analyze an intermediate form of partial integration. In particular, we consider that only two regions are integrated.

4.1 Complete Segmentation: Autarky

When the regions are in autarky they have to provide by themselves the necessary liquidity. Clearly in this case the first-best allocation cannot be decentralized by a banking system since each region (when isolated) is actually facing aggregate uncertainty.

Since in each region there is a continuum of identical banks, the decentralized allocation can be characterized in terms of the behavior of a representative bank in each region. The structure we consider is the following. Each representative bank offers fully contingent deposit contracts, specifying the amount that the depositor can withdraw at each time t contingent on the realization of the shock ω^i . A deposit contract is therefore an array

$$\delta = \left\{ y, x, \left\{ (c_t^L, c_t^H) \right\}_{t=1,2} \right\},$$

where c_t^s is the amount that a depositor can withdraw at time t if the value of the liquidity shock is ω_s , with $s = L, H$.

The allocation in autarky is given by the solution of the following problem [A]:

$$\max_{\delta} \frac{1}{2} \left[\omega_H u(c_1^H) + (1 - \omega_H) u(c_2^H) \right] + \frac{1}{2} \left[\omega_L u(c_1^L) + (1 - \omega_L) u(c_2^L) \right]$$

subject to

$$\begin{aligned}
y + x &\leq 1; \\
\omega_s c_1^s &\leq y; \quad s = L, H \\
(1 - \omega_s) c_2^s &\leq Rx + (y - \omega_s c_1^s); \quad s = L, H \\
c_t^s &\geq 0; \quad s = L, H \quad t = 1, 2.
\end{aligned}$$

The first constraint represents the feasibility constraint in $t = 0$. The second set of constraints states that in every state of the world the amount paid to depositors in $t = 1$ have to be less than the amount invested in the short asset. The third set of constraints refers to the second period. Notice that, we allow for the possibility to roll over deposits from $t = 1$ to $t = 2$. In this case the resources available to pay late consumers are given by the return on the investment in the safe asset Rx plus the resources rolled over from period 1, if any. Finally, we have the non-negativity consumption constraints.

Let

$$\delta^a = \left\{ y^a, x^a, \left\{ \left(c_t^{L(a)}, c_t^{H(a)} \right) \right\}_{t=1,2} \right\}$$

be the optimal allocation offered to consumers under autarky. We have the following

Proposition 2 *The optimal allocation δ^a satisfies*

$$c_1^{H(a)} < c_1^{L(a)} \leq c_2^{L(a)} < c_2^{H(a)}.$$

The optimal contract in autarky does not allow for roll over when the liquidity shock is high. However, when the shock is low it can be optimal to roll over resources and offer the same level of consumption in both dates. Accordingly, the inequality in Proposition 2 is strict if and only if roll-over is not optimal in state L . Otherwise, consumption is the same in both dates when the liquidity shock is low.

An immediate implication of Proposition 2 is that

$$\begin{aligned}
\omega_H u'(c_1^{H(a)}) &> \omega_L u'(c_1^{L(a)}), \\
(1 - \omega_H) u'(c_2^{H(a)}) &< (1 - \omega_L) u'(c_2^{L(a)}).
\end{aligned}$$

This means that at time one, it would be welfare improving to (slightly) reduce per capita consumption in state L to increase it in state H by the same (small) amount. However, this is not feasible in autarky since in state L any reduction of consumption at time one can be at most rolled-over at period two. Consumption at time one in state L cannot be exchanged for more first period consumption in state H .

Also notice that at time two it would be welfare improving to (slightly) reduce per capita consumption in state H to increase it in state L by the same amount. Again, such opportunity cannot be exploited in autarky. The allocation δ^a obviously gives a lower expected utility than the first-best allocation δ^* . We will call U^a the expected utility achieved under the contract δ^a .

4.2 Partial Integration with Two Regions

When only two regions (say A and B) are integrated the coinsurance mechanism provided by the interbank deposit market cannot be perfect. It would be possible to coinsure only in states S_2 and S_3 . In the other two states the coinsurance is not possible.

The probability that both banks will face the same liquidity need in $t = 1$ is $\frac{(1-p)}{2}$, while their liquidity needs will be different with probability p . Notice that $(1-p)$ is a measure of the residual aggregate uncertainty. In particular, if $p = 1$ there is no residual aggregate uncertainty, while if $p = 0$ the two banks have perfectly correlated shocks and therefore they are in the same situation as in autarky.

The structure we consider is the following. Each representative bank offers fully contingent deposit contracts, specifying the amount that the depositor can withdraw at each time t contingent on the realization of the liquidity shocks in both regions. A deposit contract is therefore an array

$$\delta = \left\{ y, x, \left\{ (c_t^{L,L}, c_t^{L,H}, c_t^{H,L}, c_t^{H,H}) \right\}_{t=1,2} \right\},$$

where $c_t^{s,s'}$ is the amount that a depositor in a given region can withdraw at time t if the liquidity shock in his region is ω_s and the liquidity shock in the integrated region is $\omega_{s'}$.

The decentralized banking system uses interbank deposits as follows:

- Each bank offers the contract δ to the consumers *and* the bank of the other region;
- Each bank deposits $k_0 \geq 0$ cents in the bank belonging to the other region and receives a deposit of $k_0 \geq 0$ from the bank of the other region.
- Each bank withdraws $k_1 \geq 0$ at time one and nothing at time two if its liquidity shock is ω_H , and it withdraws nothing at time one and $k_2 \geq 0$ at time two when its liquidity shock is ω_L .

Withdrawing of the interbank deposit is therefore contingent on the realization of the high liquidity shock ω_H . When both banks have the high liquidity shock then they have the right to withdraw the interbank deposit and no extra liquidity is available for each region since the withdrawals cancel out. When only one region has the high liquidity shock then it would withdraw the amount k_1 in $t = 1$, while the other region withdraws k_2 in $t = 2$. When both regions have the low shock, they do not withdraw at $t = 1$ but only at $t = 2$, and again the two withdrawals cancel out.

Notice that in principle $k = (k_0, k_1, k_2)$ can be contracted upon, and each bank would like to simultaneously choose δ and k to solve the following partial integration program [PI]:

$$\begin{aligned} \max_{\delta, k} \frac{1-p}{2} & \left\{ \left[\omega_H u \left(c_1^{H,H} \right) + (1 - \omega_H) u \left(c_2^{H,H} \right) \right] + \left[\omega_L u \left(c_1^{L,L} \right) + (1 - \omega_L) u \left(c_2^{L,L} \right) \right] \right\} \\ & + \frac{p}{2} \left\{ \left[\omega_H u \left(c_1^{H,L} \right) + (1 - \omega_H) u \left(c_2^{H,L} \right) \right] + \left[\omega_L u \left(c_1^{L,H} \right) + (1 - \omega_L) u \left(c_2^{L,H} \right) \right] \right\} \end{aligned}$$

subject to:

$$\begin{aligned}
x + y &\leq 1; \\
\omega_s c_1^{s,s} &\leq y; & s = L, H \\
(1 - \omega_s) c_2^{s,s} &\leq Rx + (y - \omega_s c_1^{s,s}); & s = L, H \\
\omega_H c_1^{H,L} &\leq y + k_1; \\
\omega_L c_1^{L,H} &\leq y - k_1; \\
(1 - \omega_H) c_2^{H,L} &\leq Rx + \left(y + k_1 - \omega_H c_1^{H,L} \right) - k_2; \\
(1 - \omega_L) c_2^{L,H} &\leq Rx + \left(y - k_1 - \omega_L c_1^{L,H} \right) + k_2; \\
c_t^{s,s} &\geq 0; & s = L, H \quad t = 1, 2 \\
c_t^{H,L} &\geq 0; & c_t^{L,H} \geq 0; \quad t = 1, 2.
\end{aligned}$$

The first constraint represents the feasibility constraint in $t = 0$. The second and third set of constraints represent feasibility conditions in $t = 1$ and $t = 2$, respectively, when the liquidity shocks are the same in both regions. In $t = 1$, the resources invested in the short term asset have to be sufficient to pay early consumers. In $t = 2$, the resources invested in the long term asset plus roll over, if any, are used to pay late consumers. The fourth and fifth constraints represent feasibility conditions in $t = 1$ when the liquidity shocks are asymmetric. In this case the resources available to pay early consumers are given by the investment in the short term asset plus (or minus) the interbank deposits. The sixth and seventh constraints establish feasibility conditions in $t = 2$ when the liquidity shocks are asymmetric. The resources available in $t = 2$ are given by the return of the long term asset plus the roll over, if any, plus (or minus) the interbank deposits. Finally, we have non-negativity consumption constraints in all the states of the world and all times.

The two banks solve the same problem so that they would readily agree on the conditions to apply to the interbank deposit, in particular they would both like to set an optimal k . Furthermore, notice that with $p = 0$ the above problem coincides with the autarky one, and for $p > 0$ the autarky solution is still feasible with $k = 0$. Let us call δ^{pi} the optimal contract offered in partial integration. Clearly, also δ^{pi} will give a lower expected utility than the first-best contract δ^* . Moreover, we call $U^{pi}(p)$ the expected utility achievable under the contract δ^{pi} . That is, $U^{pi}(p)$ is the value function of the partial integration problem [PI], so that $U^{pi}(0) = U^a$ represents the value function of the autarky problem [A] analyzed in Section 4.1.

It is possible to establish that consumers welfare strictly increases with any reduction in the residual aggregate uncertainty. That is,

Proposition 3 $U^{pi}(p)$ is strictly increasing in p .

To have an intuition for this result, remember that in autarky consumers cannot be insured against the liquidity shock and, therefore, first period consumption is higher in

state L than in state H , while second period consumption is higher in state H than in state L . Hence, it would be welfare improving to smooth both first and second period consumption levels across states. In a world of complete integration, i.e. with no residual aggregate uncertainty, the interbank deposit market can provide consumers with full insurance, thus achieving the first-best. In partial integration, i.e. with some residual aggregate uncertainty, consumption smoothing (across states) is only available when the integrated regions are hit by asymmetric liquidity shocks, while symmetric shocks cannot be diversified away. As a consequence, conditional on being hit by asymmetric liquidity shocks, consumers welfare can be improved, but ex-ante this happens with probability p , so that the larger p the higher consumers welfare.

Since autarky can be considered as an extreme form of partial integration with $p = 0$, we clearly have that $U^a = U^{pi}(0) < U^{pi}(p)$ for each $p > 0$. This result suggests that, whenever possible, consumers will take advantage of the coinsurance mechanism even if it is not perfect.

After having established that partial integration is always welfare improving with respect to autarky, we now analyze the allocation reached under partial integration. We first establish the following

Proposition 4 *Under partial coinsurance we have that $c_t^{H,L} = c_t^{L,H} \equiv c_t^{s,-s}$ for $t = 1, 2$. Moreover, optimal consumption can be achieved with $k_t = k_0 c_t^{s,-s}$ for $t = 1, 2$, with $k_0 = (\omega_H - \gamma)$.*

This proposition states that the interbank deposit market is used to fully coinsure against the liquidity shock provided that such coinsurance is possible, that is, provided that the two regions are not hit by the same shock. In this case the consumption level in period t is not affected by the liquidity shock and in the remainder of the paper it will be referred to as $c_t^{s,-s}$. We will also refer to states (H, H) , (L, L) and $(s, -s)$ with an obvious interpretation. An allocation is now an array

$$\delta = \left\{ y, x, \left\{ (c_t^{L,L}, c_t^{s,-s}, c_t^{H,H}) \right\}_{t=1,2} \right\}.$$

Plugging the optimal values for k_1 and k_2 into the original feasibility constraints in problem [PI], and taking into account that $c_t^{H,L} = c_t^{L,H} = c_t^{s,-s}$, the optimal allocation in partial integration is given by the solution to the following reduced partial integration problem [RPI]:

$$\begin{aligned} \max_{\delta} \quad & \frac{1-p}{2} \left[\omega_H u \left(c_1^{H,H} \right) + (1 - \omega_H) u \left(c_2^{H,H} \right) \right] + p \left[\gamma u \left(c_1^{s,-s} \right) + (1 - \gamma) u \left(c_2^{s,-s} \right) \right] \\ & + \frac{1-p}{2} \left[\omega_L u \left(c_1^{L,L} \right) + (1 - \omega_L) u \left(c_2^{L,L} \right) \right] \end{aligned}$$

subject to

$$\begin{aligned}
y + x &\leq 1; \\
\omega_s c_1^{s,s} &\leq y; & s = L, H \\
\gamma c_1^{s,-s} &\leq y; \\
(1 - \omega_s) c_2^{s,s} &\leq Rx + (y - \omega_s c_1^{s,s}); & s = L, H \\
(1 - \gamma) c_2^{s,-s} &\leq Rx + (y - \gamma c_1^{s,-s}); \\
c_t^{s,s} &\geq 0; & s = L, H \quad t = 1, 2 \\
c_t^{s,-s} &\geq 0; & t = 1, 2.
\end{aligned}$$

All the constraints have the same interpretation as in the program [PI]. Notice that, with $p = 1$, the problem [RPI] is equivalent to the problem faced by a representative bank in a fully integrated region (i.e. with no aggregate uncertainty). Given Proposition 3, this implies that $U^{pi}(p) < U^{pi}(1) = U^*$ for each $p < 1$.

Let

$$\delta^{pi} = \left\{ y^{pi}, x^{pi}, \left\{ (\bar{c}_t^{L,L}, \bar{c}_t^{s,-s}, \bar{c}_t^{H,H}) \right\}_{t=1,2} \right\}$$

be the optimal allocation offered to consumers under partial integration. We have the following

Proposition 5 *The optimal allocation with partial integration δ^{pi} satisfies*

$$\bar{c}_1^{H,H} < \bar{c}_1^{s,-s} \leq \bar{c}_1^{L,L} \leq \bar{c}_2^{L,L} \leq \bar{c}_2^{s,-s} < \bar{c}_2^{H,H}$$

Notice that all inequalities are strict if and only if rolling over is never optimal. Accordingly, roll over is never optimal when both shocks are high. Otherwise, if roll over is optimal when the shocks are asymmetric, i.e. in state $(s, -s)$, then it is also optimal when the shocks are both low, i.e. in state (L, L) . Finally, roll over can be optimal only in state (L, L) .

To conclude the analysis under partial integration, we analyze how the possibility of partial insurance affects the optimal investment in the short term asset. That is, how the optimal investment y^{pi} depends on the residual aggregate uncertainty. We have the following

Proposition 6 *Under partial integration, the optimal investment in the short term asset y^{pi} is decreasing in p if and only if $y^* < y^a$, and it is increasing in p if and only if $y^* > y^a$.*

That is, under partial integration the optimal short term investment y^{pi} always lies between the autarky level y^a and the first-best level y^* . Furthermore, any reduction in the residual aggregate uncertainty (that is, any increase in p) makes y^{pi} to move towards y^* and further away from y^a . In order to better understand Proposition 6, it is worth to analyze the following

Corollary 1 *If R is sufficiently close to 1, then the optimal investment in the short term asset $y^{pi}(p)$ is decreasing in p .*

The easiest way to grasp the intuition of the previous corollary is to assume R equal to 1.¹ On the one hand, the first-best investment in the short term asset y^* is equal to γ (see condition 3). On the other hand, in autarky the optimal investment in the short term asset y^a depends on the presence of roll over (see the Appendix for more details). Indeed, solving program [A] the first-order condition that determines y^a is

$$u'(c_1^H) + u'(c_1^L) = R [u'(c_2^H) + u'(c_2^L)].$$

Clearly, when $R = 1$ the solution without roll over $c_1^{H(a)} < c_1^{L(a)} < c_2^{L(a)} < c_2^{H(a)}$ cannot be optimal since it cannot satisfy the previous condition. Then the only solution can be the one with roll over when the liquidity shock is low, that is $c_1^{H(a)} < c_1^{L(a)} = c_2^{L(a)} < c_2^{H(a)}$. This solution implies that the short term asset is determined by the following condition

$$u'\left(\frac{y^a}{\omega_H}\right) = Ru'\left(\frac{R(1-y^a)}{1-\omega_H}\right) + (R-1)u'(R(1-y^a) + y^a).$$

Accordingly, when $R = 1$ we have $y^a = \omega_H$. Since $\gamma < \omega_H$, it follows immediately that $y^* < y^a$. By continuity, for values of R sufficiently close to 1, it still holds that $y^* < y^a$.

We are now in a position to gain the intuition for Proposition 6. We have to distinguish between two cases. The first case is when consumers in autarky find it optimal to invest more in the short term asset than the efficient investment (that is, $y^* < y^a$). In this case, consumers' main concern is to have a minimal level of consumption if they turn out to be early consumers. For this reason they are willing to give up some of the higher returns that can be achieved with the long term investment. Clearly, this is likely to be the case if R is not too large. In other words, the bad state is the one in which more people turn out to be earlier consumers. Therefore, when consumers have the opportunity to partially integrate, making coinsurance against the bad state possible, they find it optimal to reduce the short term investment. This allows consumers to take advantage of the higher long term returns.

The second case is when consumers in autarky find it optimal to invest less in the short term asset than the efficient level (that is, $y^* > y^a$). In this case, the efficient level of short term investment is (relatively) high. This may happen when the return on the long term asset is so large that consumers would like to invest efficiently in the long term asset only a (relatively) small fraction of their endowment. In this way consumers efficiently achieve a high level of consumption either as early or late consumers. Indeed, even a (relatively) small investment in the long term asset is magnified by a large R . However, in autarky

¹Formally, when $R = 1$ the optimal policy is not unique. Any investment $y \geq \omega_H$ and $y \geq \gamma$ will sustain the optimal consumption under autarky and under complete integration, respectively. However, for $R > 1$ the optimal policy is unique, and the policy we describe is the limit as R goes to one of the optimal policy.

the possibility that there might be too few early consumers prevent banks from investing too much in the short term asset. Here, the bad state is the one in which few people are earlier consumers. Therefore, when consumers have the opportunity to partially integrate, they find it optimal to increase the short term investment. This allows to guarantee also early consumers a sufficiently high level of consumption.

It is worth to show numerically under which conditions the two cases arise. Let us assume a CRRA utility function with relative risk aversion equal to θ , and different values for the liquidity shocks ω_L and ω_H . Figure 1 shows for which values of R and $\theta > 1$ the condition $y^* < y^a$ is satisfied.² Considering the values for the liquidity shocks as $\omega_L = 0.5$ and $\omega_H = 0.7$, which imply $\gamma = 0.6$ and $\omega_H - \omega_L = 0.2$, the condition $y^* < y^a$ is satisfied in the region above the line labeled $\gamma = 0.6$. A similar interpretation can be given to the other lines.³ It is clear that for reasonable values of the return of the long term asset (that is, $R < 1.5$) the condition is always satisfied. For higher values of R (that is, $1.5 < R < 2.5$), then we need a sufficiently high risk aversion (i.e., $\theta > 3$) for the condition $y^* < y^a$ to be satisfied.

Figure 2 assumes a higher variance for the liquidity shocks but otherwise is identical to Figure 1. In this case, the region where the condition $y^* < y^a$ arises is even larger. Indeed, for $R < 2.2$ the condition $y^* < y^a$ always holds and for $2.2 < R < 3$ it holds for $\theta > 2$.

In the remainder of the paper we will mainly focus on the case in which $y^* < y^a$. We consider this case to be more natural to study the partial coinsurance mechanism. Indeed, it seems plausible that coinsurance mechanisms are set to face possible liquidity shortage. In the rationale of our model, this means that the bad state is represented by the one with more early consumers. That is, the coinsurance mechanism is a device to decrease liquidity holding with respect to the autarky situation.

5 Financial Integration and the Depth of Systemic Crises

In the last section we have established that, whenever possible, the regions find it convenient to coinsure against the liquidity shocks. This result is driven by the fact that the uncertainty of liquidity needs is the highest in autarky and it is reduced when the residual aggregate uncertainty becomes lower.

This is easy to verify. Let us compute the variance of liquidity shock in autarky. Given

²For values of $\theta < 1$ the condition $y^* < y^a$ is always satisfied in our numerical examples (available upon request).

³Also in this case, for values of $\gamma \leq 0.5$ the condition $y^* < y^a$ is always satisfied in our numerical examples (available upon request).

that the expected liquidity shock is $\frac{1}{2}\omega_H + \frac{1}{2}\omega_L = \gamma$, the variance is

$$Var_{\omega}^a = \left(\frac{\omega_H - \omega_L}{2} \right)^2.$$

Under partial integration the expected liquidity shock is

$$\frac{1-p}{2}\omega_H + p\gamma + \frac{1-p}{2}\omega_L = \gamma,$$

and the variance is

$$Var_{\omega}^{pi}(p) = \frac{1-p}{2}[(\omega_H - \gamma)^2 + (\omega_L - \gamma)^2] = (1-p)Var_{\omega}^a.$$

Clearly, if $p = 1$ (no residual aggregate uncertainty) the variance of the liquidity shocks is zero (as in the first-best solution) and if $p = 0$ the variance in partial integration is the same as in autarky (there is no reduction in residual aggregate uncertainty). The variance of liquidity shocks under partial coinsurance is strictly decreasing in p .

In this section we analyze the implications of partial integration for the depth of the systemic crisis. We use two measures: the first is the cost of liquidity, i.e. the interest rate charged by banks in the interbank market; the second is the real effect, i.e. the variance and skewness of the optimal consumption profile. That is, once the regions have the possibility to reduce only partially the residual aggregate uncertainty, what are the implications for the cost of liquidity and the optimal consumption distribution?

5.1 The Cost of Liquidity

In the present model the liquidity of a given bank can be defined as the amount of resources it has available at $t = 1$. In the first place, this means the amount invested in the short asset y , but this amount can be increased at $t = 1$ by withdrawing k_1 from the interbank deposit market. In this way the bank increases the liquidity at its disposal but loses the opportunity of withdrawing k_2 at $t = 2$ from the same interbank market.⁴

Considering partial integration, the previous observation naturally leads to the following

Definition 1 *Under partial integration, the cost of liquidity at $t = 1$ and state (s_1, s_2) , with $s_i \in \{H, L\}$, $i = 1, 2$, is equal to*

$$CL^{s_1, s_2} = \frac{k_2}{k_1} = \frac{k_0 c_2^{s_1, s_2}}{k_0 c_1^{s_1, s_2}} = \frac{c_2^{s_1, s_2}}{c_1^{s_1, s_2}}.$$

⁴Notice that the cost of liquidity defined in the text represents the cost of liquidity in $t = 1$. Ex ante, i.e. in $t = 0$, the opportunity cost of investing in the short asset coincides with the missed return on the long asset (i.e., R). In a sense, this ex ante opportunity cost of liquidity is completely determined by the technology, and it is independent of the regions' degree of integration.

The interpretation is straightforward. The ratio $c_2^{s_1, s_2} / c_1^{s_1, s_2}$ represents the gross interest rate on the interbank deposit market. That is, the opportunity cost that the bank faces in gathering liquidity in $t = 1$ in terms of lost second period resources. In other words, the cost of (increasing) liquidity in $t = 1$ is defined in terms of the amount of resources given up for consumption in $t = 2$. Notice that the incentive compatibility (IC) constraint $c_1^{s_1, s_2} \geq c_2^{s_1, s_2}$ implies $CL^{s_1, s_2} \geq 1$. Furthermore, if there is some positive roll over in state (s_1, s_2) then the (IC) constraint binds and the above definition implies a value for the cost of liquidity equal to one. Indeed, whenever banks are rolling over liquidity in $t = 1$ it is clear that there is an excess of liquidity and its cost drops to the minimum. However, if there is no roll over in state (s_1, s_2) then the (IC) constraint is slack and we have $CL^{s_1, s_2} > 1$.

Under complete integration there is no residual aggregate uncertainty, so there is only one state of the world in which there is exactly a fraction γ of early consumers. In this case we have the following

Definition 2 *Under complete integration, the cost of liquidity at $t = 1$ is equal to*

$$CL^* = \frac{c_2^*}{c_1^*} = \frac{\gamma}{1 - \gamma} \frac{(1 - y^*)}{y^*} R. \quad (5)$$

Since the incentive compatibility condition in complete integration is not binding ($c_2^* > c_1^*$), the cost of liquidity under complete integration is always constant and greater than one, i.e. $CL^* > 1$. Notice also that the relationship between the cost of liquidity, as defined above, and the aggregate fraction of early consumers γ is not necessarily monotonic. In fact, implicitly differentiating condition (3) we can obtain $\frac{dy^*}{d\gamma} > 0$, so that the sign of $\frac{dCL^*}{d\gamma}$ cannot be determined. Intuitively, the larger the fraction of early consumers, the larger the investment in the short asset. Hence, in $t = 1$, the greater γ the higher the need of liquid resources, which points toward a higher cost of liquidity. However, a greater γ also implies a larger amount of liquid resources already available before getting to the interbank deposit market, which points toward a smaller cost of liquidity. The overall effect is then ambiguous.

Under partial integration the cost of liquidity in state (s_1, s_2) depends on the degree of integration, that is, on the value of p . We have the following

Proposition 7 *Assume $y^* < y^a$, then the cost of liquidity under partial integration is strictly increasing in p in state (H, H) , and it is non decreasing in p in any other state (s_1, s_2) .*

The intuition is the following. When $y^* < y^a$, the investment in the short liquid asset decreases with p so that, ceteris paribus, there will be an increased need of resources in $t = 1$ in any state with no roll over, which induces a higher cost of liquidity in any such state.

In autarky, since there is no interbank deposit market available, to increase liquidity beyond the short asset y is not even feasible. In this case the cost of liquidity can be considered to be infinite. This is quite intuitive since in this case it is impossible to smooth out the (maximum) variance of the idiosyncratic liquidity shocks. Consequently, both under perfect integration and partial integration (i.e., whenever the interbank deposit market is feasible) the cost of liquidity is clearly lower than in autarky. However, it is not clear how the cost of liquidity is related under complete integration and partial integration. In particular, is the cost of liquidity under partial integration lower or higher than under complete integration? The following result provides an answer to this question.

Proposition 8 *Assume $y^* < y^a$, then the cost of liquidity under partial and full integration are ordered as follows:*

$$1 \leq CL^{L,L} \leq CL^{s,-s} < CL^* < CL^{H,H} \quad \forall p \in (0, 1).$$

To have an intuition for the result consider that, under the maintained assumption $y^* < y^a$, banks invest more resources in the short liquid asset if they are partially integrated than if they are fully integrated. They do so because they face some, possibly small, probability of a systemic high liquidity need, that is the state (H, H) . However, in any other state, this extra amount of liquid resources turns out to be excessive (with respect to the first best) and drives down the cost of liquidity. When, on the contrary, the realized state is (H, H) the liquidity is however not enough and banks in both regions try to gather more resources on the interbank deposit market thus pushing up the cost of liquidity with respect the complete integration case.

It is interesting to show by means of examples how the costs of liquidity are determined. Again, let us assume a CRRA utility function with relative risk aversion equal to θ . In Figure 3 we assume $\theta = 5$, $R = 1.4$, $\omega_L = 0.1$ and $\omega_H = 0.8$. In this case, the cost of liquidity under complete integration is slightly higher than one. The cost of liquidity under partial integration when the state is either (L, L) or $(s, -s)$ is equal to 1 since optimality requires roll over in both states. Under partial integration the cost of liquidity is then very close to the one under complete integration. However, in state (H, H) the cost of liquidity can become very high when residual aggregate uncertainty is very low. The cost of liquidity can be more than five times higher then the cost of liquidity under complete integration (or partial integration without crisis).

In Figure 4 we assume $\theta = 1.5$, $R = 1.8$, $\omega_L = 0.6$ and $\omega_H = 0.9$. In this case the cost of liquidity under complete integration is 1.5. The cost of liquidity under partial integration is equal to 1 in state (L, L) since roll over is optimal. In state $(s, -s)$ roll over becomes not optimal and the cost of liquidity in this case increases with p , converging to the cost of liquidity under complete integration for $p \rightarrow 1$. The cost of liquidity in state (H, H) is well above the others for all the values of p . It ranges from 2 to 3 times the cost of liquidity under complete integration.

5.2 The Distribution of Optimal Consumption

In order to assess the real consequences of partial integration, we analyze the variance and the skewness of optimal consumption. We focus our attention on the consumption profile in $t = 1$ since this is the date where the systemic crisis can occur.

5.2.1 Consumption Variance

In the first-best allocation, consumption uncertainty is completely eliminated through the interbank deposit market, while in autarky and partial integration perfect coinsurance is not feasible and consumption variance is unavoidable. Of course, as the residual aggregate uncertainty vanishes, i.e. as p approaches one, the consumption variance in partial integration also vanishes. However, we show that if the residual aggregate uncertainty under partial integration is large enough, consumption variance could be larger in partial integration than in autarky.

Let Var^a be the consumption variance in autarky. With some simple algebra it is possible to obtain

$$Var^a = \left(\frac{c_1^{L(a)} - c_1^{H(a)}}{2} \right)^2.$$

Similarly, let $Var^{pi}(p)$ be the consumption variance under partial integration, when the probability of asymmetric shocks is p . Let us denote with $\bar{c}_1^{s,s}$ the average consumption in $t = 1$ when the integrated regions are hit by the same liquidity shock, that is

$$\bar{c}_1^{s,s} = \frac{\bar{c}_1^{H,H} + \bar{c}_1^{L,L}}{2}.$$

Then the consumption variance in partial integration can be written as

$$Var^{pi}(p) = (1 - p) \left[\left(\frac{\bar{c}_1^{L,L} - \bar{c}_1^{H,H}}{2} \right)^2 + 2p \left(\frac{\bar{c}_1^{s,s} - \bar{c}_1^{s,-s}}{2} \right)^2 \right].$$

This expression has a useful interpretation. Possible values of first period consumption can be divided into two groups: consumption in case of undiversifiable liquidity shocks, $\bar{c}_1^{L,L}$ and $\bar{c}_1^{H,H}$ (with average $\bar{c}_1^{s,s}$), and consumption in case of diversifiable liquidity shock, $\bar{c}_1^{s,-s}$ (clearly with average $\bar{c}_1^{s,-s}$). Total consumption variance then depends on both the within-groups variance

$$\left(\frac{\bar{c}_1^{L,L} - \bar{c}_1^{H,H}}{2} \right)^2$$

and on the between-groups variance

$$\left(\frac{\bar{c}_1^{s,s} - \bar{c}_1^{s,-s}}{2} \right)^2.$$

Furthermore, note that we have $Var^{pi}(0) = Var^a$. This immediately follows from the fact that, when $p = 0$, the allocations in autarky and partial integration coincide (i.e., $c_1^{L(a)} = \bar{c}_1^{L,L}$ and $c_1^{H(a)} = \bar{c}_1^{H,H}$). On the other hand, we have $Var^{pi}(1) = 0$ that is equal to the first-best consumption variance. However, for $p \in (0, 1)$ consumption variance in partial integration can be higher or lower than the variance in autarky. We have the following

Proposition 9 *If $y^* < y^a$, and rolling over is optimal in both states (L, L) and $(s, -s)$ for p around 0, then it exists a threshold p^* such that*

$$Var^{pi}(p) > Var^a \text{ for } p \in (0, p^*).$$

Note that both conditions are satisfied for R sufficiently close to 1. To have an intuition for this result, notice that for p close to zero the within-group component of total variance dominates and, if rolling over is optimal in both states (L, L) and $(s, -s)$, it is decreasing in y . Furthermore, if $y^* < y^a$ Proposition 6 implies that $y(p)$ decreases in p so that, for p small enough, the larger p , the smaller $y(p)$, the larger the variance. This result shows that when financial integration leaves a sufficiently large residual aggregate uncertainty, then the two regions would experience higher consumption variance than under autarky. However, it is optimal to expose the system to this higher consumption variability since the partial integration implies also higher expected utility.

Figures 5 and 6 show two cases of Proposition 9 where, under the assumption of a CRRA utility function with relative risk aversion equal to θ and plausible parameter values, the standard deviation of consumption under partial integration can be higher than the one under autarky.

In Figure 5 we assume $\theta = 5$, $R = 1.4$, and $\omega_H - \omega_L = 0.7$. In this case, the volatility of consumption under partial integration is always higher than the one in autarky (except when the residual aggregate uncertainty is close to zero). It is interesting to notice that the maximum volatility is obtained for a relatively high value of partial integration (i.e., $p = 0.9$). At the peak the magnitude of volatility under partial integration is 4 times the one in autarky. In Figure 6 we assume $\theta = 1.5$, $R = 1.8$, and $\omega_H - \omega_L = 0.3$. In this case, volatility of consumption under partial integration is higher than the one in autarky until $p = 0.65$. The order of magnitude is lower.

5.2.2 Consumption Skewness

The skewness is a measure of the symmetry of a certain distribution. By definition, a symmetric distribution has zero skewness while, roughly speaking, negative skewness indicates that the left tail is longer while the mass of the distribution is concentrated on the right of its support. The opposite happens for a distribution with positive skewness. Notice that negative skewness characterizes consumption distributions that allow for particularly low values of consumption with some small probability.

Given a random variable X with mean μ , standard deviation σ , and third moment

$$M_3 = \sum_x \Pr(x)(x - \mu)^3,$$

we consider, the standardized third moment as a measure for the skewness sk .⁵ That is,

$$sk = \frac{M_3}{\sigma^3}.$$

In autarky, first period consumption is distributed symmetrically, indeed it is either $c_1^{H(a)}$ with probability 1/2 or $c_1^{L(a)}$ with probability 1/2. Accordingly, its skewness is zero, that is $sk^a = 0$. In the first-best allocation, consumption is not random so that its skewness is zero as well, that is $sk^* = 0$. In partial integration, consumption can assume the values $\bar{c}_1^{H,H}$, $\bar{c}_1^{s,-s}$ or $\bar{c}_1^{L,L}$ with probabilities $\frac{1-p}{2}$, p and $\frac{1-p}{2}$, respectively. Let $sk^{pi}(p)$ denote the consumption skewness in partial integration as a function of p . In order to provide a useful characterization of its sign, we establish the following

Lemma 1. *The quantity $sk^{pi}(p)$ has the same sign as $\left[\left(\bar{c}_1^{L,L} - \bar{c}_1^{s,-s} \right) - \left(\bar{c}_1^{s,-s} - \bar{c}_1^{H,H} \right) \right]$, that is*

$$sk^{pi}(p) \begin{cases} < 0 & \text{if } \left(\bar{c}_1^{L,L} - \bar{c}_1^{s,-s} \right) < \left(\bar{c}_1^{s,-s} - \bar{c}_1^{H,H} \right) \\ = 0 & \text{if } \left(\bar{c}_1^{L,L} - \bar{c}_1^{s,-s} \right) = \left(\bar{c}_1^{s,-s} - \bar{c}_1^{H,H} \right) \\ > 0 & \text{if } \left(\bar{c}_1^{L,L} - \bar{c}_1^{s,-s} \right) > \left(\bar{c}_1^{s,-s} - \bar{c}_1^{H,H} \right). \end{cases}$$

In principle the consumption skewness can be either zero, positive or negative depending on the model's parameters. However, notice that if roll over is optimal in both states (L, L) and $(s, -s)$, then $\bar{c}_1^{L,L} = \bar{c}_1^{s,-s}$ and, therefore

$$0 = \left(\bar{c}_1^{L,L} - \bar{c}_1^{s,-s} \right) < \left(\bar{c}_1^{s,-s} - \bar{c}_1^{H,H} \right)$$

which implies that the skewness is surely negative.

On the other hand, if roll over is never optimal, it is a matter of simple algebra to check that

$$\frac{\omega_H - \omega_L y^{pi}}{\omega_H + \omega_L \omega_L} = \left(\bar{c}_1^{L,L} - \bar{c}_1^{s,-s} \right) > \left(\bar{c}_1^{s,-s} - \bar{c}_1^{H,H} \right) = \frac{\omega_H - \omega_L y^{pi}}{\omega_H + \omega_L \omega_H}$$

so that the skewness is surely positive. Finally, if roll over is optimal in state (L, L) but not in state $(s, -s)$, then the consumption skewness can be either positive or negative. We have the following

Proposition 10 *Assume $y^* < y^a$, and let $\tilde{y} = \frac{\gamma \omega_H R}{\gamma \omega_H R + 2\omega_H - \gamma(1 + \omega_H)}$, then it exists a $\tilde{p} \in (0, 1)$ such that*

⁵Alternative, simpler measures are provided by the Pearson skewness coefficients $\frac{3(\mu - mode)}{\sigma}$ and $\frac{3(\mu - median)}{\sigma}$. Notice that if a distribution has a negative skewness according to a certain measures, then it also has a negative skewness according to other measures. Clearly, the absolute value will be different. Our results on consumption skewness refers to its sign so that they are independent of the skewness measure.

1. if $\tilde{y} \leq y^* < y^a$ then $sk^{pi}(p) < 0$ for each p
2. if $y^* < \tilde{y} < y^a$ then $sk^{pi}(p) \begin{cases} < 0 & \text{if } p < \tilde{p} \\ = 0 & \text{if } p = \tilde{p} \\ > 0 & \text{if } p > \tilde{p} \end{cases}$
3. if $y^* < y^a \leq \tilde{y}$ then $sk^{pi}(p) > 0$ for each p .

Also for the skewness we established conditions where the partial integration implies a negative skewness in the consumption distribution. In particular, this will be always the case when roll over is optimal in both the state (L, L) and $(s, -s)$. When roll over is optimal only when the two liquidity shocks are low, the skewness will be negative if partial integration leaves a sufficiently large residual aggregate uncertainty.

To illustrate numerically when a negative skewness arises, we assume again a CRRA utility with relative risk aversion θ . In many examples it turns out that, by increasing θ and holding fixed the other parameters, one can move from case 3 to case 2 and then from case 2 to case 1 in Proposition 10. That is, for θ small enough the model behaves according to case 3 but when θ reaches a certain threshold then the model switches to case 2. For larger values of θ the model stays in case 2 until another threshold is hit that makes it switch to case 1. Table 1 shows these thresholds in some numerical examples. It is evident that for $R \leq 3$ a value of $\theta > 3$ guarantees that case 1 will prevail (that is, the skewness will be negative).

Propositions 9 and 10 immediately imply the following

Corollary 2 *Assume $y^* < y^a$, $p < p^*$, and rolling over being optimal in state (L, L) and $(s, -s)$. Then $sk^{pi}(p) < 0$ and $Var^{pi}(p) > Var^a$.*

Table 2 and 3 contain consumption standard deviation and skewness in several examples. Note that the constant standard deviation in autarky corresponds to the value of $p = 0$. Independently of the residual aggregate uncertainty, the skewness is always negative under case 1 and always positive under case 3. A sufficiently high residual aggregate uncertainty (i.e., a sufficiently low p) guarantees a negative skewness also under case 2.

6 Conclusions

In this paper we show that partial financial integration can be ultimate cause of extreme events. Partial integration is always welfare improving with respect the autarky as long as it helps in reducing aggregate uncertainty. This implies that banks optimally reduce the amount of liquidity holding, which is equivalent to say that liquidity holding in autarky is higher than liquidity holding under complete integration. However, less liquidity holding opens up the possibility to extreme events. Indeed, when the crisis occurs, under partial

integration the cost of liquidity can be unusually high. Moreover, the optimal consumption distribution under partial integration is characterized by higher volatility than under autarky. Also, negative consumption skewness can be displayed. We have shown that, even for low amount of residual aggregate uncertainty (i.e., very low probability of systemic crisis), all these effects can be quite remarkable and the systemic crisis can be deep indeed.

Appendix

Proof of Proposition 1. In a competitive equilibrium the deposit contract offered by the representative banks maximizes the ex ante expected utility of the consumers. All we need to show is that the constraints faced by the representative banks, with the help of the interbank deposit market, are the same as the constraints faced by the social planner both in $t = 1$ and $t = 2$.

The region with high liquidity shock has the following budget constraints in $t = 1$ and $t = 2$:

$$\omega_H c_1 \leq y + (\omega_H - \gamma)c_1,$$

and

$$(1 - \omega_H)c_2 + (\omega_H - \gamma)c_2 \leq Rx.$$

Both constraints are the same of the social planner problem. The region with low liquidity shock has the following budget constraints in $t = 1$ and $t = 2$:

$$\omega_L c_1 + (\omega_H - \gamma)c_1 \leq y,$$

and

$$(1 - \omega_L)c_2 \leq Rx + (\omega_H - \gamma)c_2.$$

Since $\omega_H - \gamma = \gamma - \omega_L$, also this region offer first-best allocation since the constraints are the same of the social planner. ■

Proof of Proposition 2. Consider the autarky problem [A]. Optimality requires $x + y = 1$ (no waste of resources). The problem then can be simplified as follows:

$$\max_{\delta} \frac{1}{2} [\omega_H u(c_1^H) + (1 - \omega_H) u(c_2^H)] + \frac{1}{2} [\omega_L u(c_1^L) + (1 - \omega_L) u(c_2^L)]$$

subject to

$$\begin{aligned} \omega_L c_1^L &\leq y; \\ \omega_H c_1^H &\leq y; \\ (1 - \omega_H) c_2^H &\leq R(1 - y) + y - \omega_H c_1^H; \\ (1 - \omega_L) c_2^L &\leq R(1 - y) + y - \omega_L c_1^L; \\ c_t^s &\geq 0; \quad s = L, H \quad t = 1, 2. \end{aligned}$$

Inada conditions guarantee that $c_t^s > 0 \forall s$ and $\forall t$, and that $y \in (0, 1)$.

Let μ_t^s be the multiplier for the feasibility constraint at time t in state s . First order conditions for an interior solution are then:

$$\frac{1}{2}u'(c_1^H) = \mu_1^H + \mu_2^H \quad (6)$$

$$\frac{1}{2}u'(c_1^L) = \mu_1^L + \mu_2^L \quad (7)$$

$$\frac{1}{2}u'(c_2^H) = \mu_2^H \quad (8)$$

$$\frac{1}{2}u'(c_2^L) = \mu_2^L \quad (9)$$

$$u'(c_1^H) + u'(c_1^L) = R[u'(c_2^H) + u'(c_2^L)]. \quad (10)$$

It immediately follows that $\mu_2^s > 0$ so that second period feasibility constraints are binding in both states. Moreover, first period feasibility constraints cannot be both slack, otherwise it would be possible to increase expected utility by investing slightly more in the long term asset.

Assume that $\omega_H c_1^H < y$ so that $\mu_1^H = 0$, and $\omega_L c_1^L = y$. Therefore, it follows that $c_1^L = \frac{y}{\omega_L} > \frac{y}{\omega_H} > c_1^H$, and then also

$$\mu_2^H = \frac{1}{2}u'(c_1^H) > \frac{1}{2}u'(c_1^L) = \mu_1^L + \mu_2^L,$$

which implies $\mu_2^H > \mu_2^L$ and then $c_2^H < c_2^L$. However, from feasibility constraints it also results that

$$c_2^H = \frac{R(1-y) + (y - \omega_H c_1^H)}{(1 - \omega_H)} > \frac{R(1-y)}{(1 - \omega_H)} > \frac{R(1-y)}{(1 - \omega_L)} = c_2^L$$

which is a contradiction. As a consequence, roll-over is never optimal in state H .

Roll-over could be optimal in state L . From (7) and (9) we have

$$u'(c_1^L) - 2\mu_1^L = u'(c_2^L)$$

then $c_1^L \leq c_2^L$ (so incentive compatibility is satisfied when the shock is low). We have two cases to consider:

- $\mu_1^L > 0$ (solution without roll over in state L). Then we have $c_1^L = \frac{y}{\omega_L} > \frac{y}{\omega_H} = c_1^H$, and this implies that $c_2^H = \frac{R(1-y)}{1-\omega_H} > \frac{R(1-y)}{1-\omega_L} = c_2^L$. So we have $c_1^H < c_1^L < c_2^L < c_2^H$ and y is determined by (10) that becomes

$$u'\left(\frac{y}{\omega_H}\right) + u'\left(\frac{y}{\omega_L}\right) = R\left[u'\left(\frac{R(1-y)}{1-\omega_L}\right) + u'\left(\frac{R(1-y)}{1-\omega_H}\right)\right]. \quad (11)$$

- $\mu_1^L = 0$ (solution with roll over in state L). In this case we have $c_1^L = c_2^L < \frac{y}{\omega_L}$, this implies that $c_2^L = \frac{R(1-y)+y-\omega_L c_1^L}{1-\omega_L} \implies c_2^L = R(1-y) + y$. From (10) we also have that

$$\frac{1}{2}u'(c_1^H) = u'(c_2^H)R + u'(c_2^L)(R-1) > u'(c_2^H) \implies c_1^H < c_2^H,$$

and that

$$\frac{1}{2}u'(c_1^H) = u'(c_2^H)R + u'(c_2^L)(R-1) > u'(c_2^L) \implies c_1^H < c_2^L = c_1^L.$$

Finally, notice that

$$\begin{aligned} c_2^H > c_1^H &\implies \frac{R(1-y)}{1-\omega_H} > \frac{y}{\omega_H} \implies R(1-y)\frac{\omega_H}{1-\omega_H} > y \implies R(1-y)\left[\frac{1}{1-\omega_H} - 1\right] > y \\ &\implies \frac{R(1-y)}{1-\omega_H} > R(1-y) + y \implies c_2^H > c_2^L. \end{aligned}$$

So we have $c_1^H < c_1^L = c_2^L < c_2^H$ and y is determined by (10) that becomes

$$u'\left(\frac{y}{\omega_H}\right) = Ru'\left(\frac{R(1-y)}{1-\omega_H}\right) + (R-1)u'(R(1-y) + y). \quad (12)$$

■

Proof of Proposition 3. Let $\{\tilde{c}_t\}_{t=1,2} = \left\{(\tilde{c}_t^{L,L}, \tilde{c}_t^{L,H}, \tilde{c}_t^{H,L}, \tilde{c}_t^{H,H})\right\}_{t=1,2}$ be the optimal consumption levels for some $p > 0$. Let us show that the following inequality

$$\begin{aligned} &\left[\omega_H u\left(\tilde{c}_1^{H,H}\right) + (1-\omega_H)u\left(\tilde{c}_2^{H,H}\right)\right] + \left[\omega_L u\left(\tilde{c}_1^{L,L}\right) + (1-\omega_L)u\left(\tilde{c}_2^{L,L}\right)\right] \geq \\ &\left[\omega_H u\left(\tilde{c}_1^{H,L}\right) + (1-\omega_H)u\left(\tilde{c}_2^{H,L}\right)\right] + \left[\omega_L u\left(\tilde{c}_1^{L,H}\right) + (1-\omega_L)u\left(\tilde{c}_2^{L,H}\right)\right] \end{aligned}$$

cannot hold. In fact, if the inequality is strict the alternative consumption plan

$$\left\{\left(\tilde{c}_t^{H,H}, \tilde{c}_t^{H,H}, \tilde{c}_t^{L,L}, \tilde{c}_t^{L,L}\right)\right\}_{t=1,2},$$

which is surely feasible with $k = 0$, strictly improves consumers' utility when the probability of asymmetric shocks is p , contradicting that \tilde{c} is optimal.

In case of equality, both $\left(\tilde{c}_1^{H,H}, \tilde{c}_1^{L,L}, \tilde{c}_2^{H,H}, \tilde{c}_2^{L,L}\right)$ and $\left(\tilde{c}_1^{H,L}, \tilde{c}_1^{L,H}, \tilde{c}_2^{H,L}, \tilde{c}_2^{L,H}\right)$ must be the autarky level of consumption, that is

$$\{\tilde{c}_t\}_{t=1,2} = \left\{\left(c_t^{L(a)}, c_t^{L(a)}, c_t^{H(a)}, c_t^{H(a)}\right)\right\}_{t=1,2}.$$

To see this, note that if a generic consumption profile

$$\{c_t\}_{t=1,2} = \left\{\left(c_t^{L,L}, c_t^{L,H}, c_t^{H,L}, c_t^{H,H}\right)\right\}_{t=1,2}$$

is feasible in the partial integration problem for some k , the consumption levels

$$\left\{ (c_t^{H,H}, c_t^{L,L}) \right\}_{t=1,2}$$

are feasible also in the autarky problem and, therefore, the quantity

$$\frac{1}{2} \left[\omega_H u \left(\tilde{c}_1^{H,H} \right) + (1 - \omega_H) u \left(\tilde{c}_2^{H,H} \right) \right] + \frac{1}{2} \left[\omega_L u \left(\tilde{c}_1^{L,L} \right) + (1 - \omega_L) u \left(\tilde{c}_2^{L,L} \right) \right]$$

is bounded above by the autarky utility U^a . Furthermore, U^a is only achieved with autarky consumption levels (i.e. the solution to the autarky problem is unique) so that if

$$\begin{aligned} & \left[\omega_H u \left(\tilde{c}_1^{H,H} \right) + (1 - \omega_H) u \left(\tilde{c}_2^{H,H} \right) \right] + \left[\omega_L u \left(\tilde{c}_1^{L,L} \right) + (1 - \omega_L) u \left(\tilde{c}_2^{L,L} \right) \right] = \\ & \left[\omega_H u \left(\tilde{c}_1^{H,L} \right) + (1 - \omega_H) u \left(\tilde{c}_2^{H,L} \right) \right] + \left[\omega_L u \left(\tilde{c}_1^{L,H} \right) + (1 - \omega_L) u \left(\tilde{c}_2^{L,H} \right) \right] \end{aligned}$$

and consumption levels are not at the autarky level, consumers welfare could be strictly increased by offering them

$$\left\{ (c_t^{L(a)}, c_t^{L(a)}, c_t^{H(a)}, c_t^{H(a)}) \right\}_{t=1,2},$$

which is feasible with $k = 0$.

This shows that, if the previous equality holds, it must be that

$$\left\{ \tilde{c}_t \right\}_{t=1,2} = \left\{ (c_t^{L(a)}, c_t^{L(a)}, c_t^{H(a)}, c_t^{H(a)}) \right\}_{t=1,2}.$$

However, if this is the case, since the autarky consumption levels are feasible with $k = 0$, the alternative consumption plan c^ε with

$$c_1^\varepsilon = (c_1^{H(a)}, c_1^{H(a)} + \frac{\varepsilon}{\omega_H}, c_1^{L(a)}, c_1^{L(a)} - \frac{\varepsilon}{\omega_L})$$

and

$$c_2^\varepsilon = (c_2^{H(a)}, c_2^{H(a)} - \frac{\varepsilon}{1 - \omega_L}, c_2^{L(a)}, c_2^{L(a)} + \frac{\varepsilon}{1 - \omega_H})$$

is feasible with y^a and $k = (\varepsilon, \varepsilon, \varepsilon)$.

Furthermore, with respect to autarky the consumption plan c^ε generates a variation in utility given by:

$$\begin{aligned} & \omega_H \left[u \left(c_1^{H(a)} + \frac{\varepsilon}{\omega_H} \right) - u \left(c_1^{H(a)} \right) \right] + \omega_L \left[u \left(c_1^{L(a)} - \frac{\varepsilon}{\omega_L} \right) - u \left(c_1^{L(a)} \right) \right] + \\ & (1 - \omega_H) \left[u \left(c_2^{H(a)} - \frac{\varepsilon}{1 - \omega_H} \right) - u \left(c_1^{H(a)} \right) \right] + (1 - \omega_L) \left[u \left(c_1^{L(a)} + \frac{\varepsilon}{1 - \omega_L} \right) - u \left(c_1^{L(a)} \right) \right] \end{aligned}$$

that for ε small is approximately equal to

$$\varepsilon \left\{ \left[u'(c_1^{H(a)}) - u'(c_1^{L(a)}) \right] + \left[u'(c_2^{L(a)}) - u'(c_2^{L(a)}) \right] \right\} > 0,$$

so that c^ε is an improvement over autarky. This is a contradiction and, therefore, it must be that

$$\begin{aligned} & \left[\omega_H u \left(\tilde{c}_1^{H,H} \right) + (1 - \omega_H) u \left(\tilde{c}_2^{H,H} \right) \right] + \left[\omega_L u \left(\tilde{c}_1^{L,L} \right) + (1 - \omega_L) u \left(\tilde{c}_2^{L,L} \right) \right] < \\ & \left[\omega_H u \left(\tilde{c}_1^{H,L} \right) + (1 - \omega_H) u \left(\tilde{c}_2^{H,L} \right) \right] + \left[\omega_L u \left(\tilde{c}_1^{L,H} \right) + (1 - \omega_L) u \left(\tilde{c}_2^{L,H} \right) \right]. \end{aligned}$$

Consider now $p' > p$, and notice that feasibility constraints are not affected by p so that $\{\tilde{c}_t\}_{t=1,2}$ is feasible also with p' . Furthermore,

$$\begin{aligned} U(p') & \geq \frac{1-p'}{2} \left\{ \left[\omega_H u \left(\tilde{c}_1^{H,H} \right) + (1 - \omega_H) u \left(\tilde{c}_2^{H,H} \right) \right] + \left[\omega_L u \left(\tilde{c}_1^{L,L} \right) + (1 - \omega_L) u \left(\tilde{c}_2^{L,L} \right) \right] \right\} + \\ & \frac{p'}{2} \left\{ \left[\omega_H u \left(\tilde{c}_1^{H,L} \right) + (1 - \omega_H) u \left(\tilde{c}_2^{H,L} \right) \right] + \left[\omega_L u \left(\tilde{c}_1^{L,H} \right) + (1 - \omega_L) u \left(\tilde{c}_2^{L,H} \right) \right] \right\} > \\ & \frac{1-p}{2} \left\{ \left[\omega_H u \left(\tilde{c}_1^{H,H} \right) + (1 - \omega_H) u \left(\tilde{c}_2^{H,H} \right) \right] + \left[\omega_L u \left(\tilde{c}_1^{L,L} \right) + (1 - \omega_L) u \left(\tilde{c}_2^{L,L} \right) \right] \right\} + \\ & \frac{p}{2} \left\{ \left[\omega_H u \left(\tilde{c}_1^{H,L} \right) + (1 - \omega_H) u \left(\tilde{c}_2^{H,L} \right) \right] + \left[\omega_L u \left(\tilde{c}_1^{L,H} \right) + (1 - \omega_L) u \left(\tilde{c}_2^{L,H} \right) \right] \right\} = U(p). \end{aligned}$$

■

Proof of Proposition 4. Consider the partial integration program [PI]. Let $\mu_t^{s,s'}$ be the multiplier associated with the feasibility constraint at time t and state s, s' . Among the first order conditions of the problem we have the following:

$$\begin{aligned} c_1^{H,L} : & \quad \frac{p}{2} u'(c_1^{H,L}) = \mu_1^{H,L} + \mu_2^{H,L} \\ c_1^{L,H} : & \quad \frac{p}{2} u'(c_1^{L,H}) = \mu_1^{L,H} + \mu_2^{L,H} \\ c_2^{H,L} : & \quad \frac{p}{2} u'(c_2^{H,L}) = \mu_2^{H,L} \\ c_2^{L,H} : & \quad \frac{p}{2} u'(c_2^{L,H}) = \mu_2^{L,H} \\ k_1 : & \quad \mu_1^{H,L} = \mu_1^{L,H} \\ k_2 : & \quad \mu_2^{H,L} = \mu_2^{L,H}. \end{aligned}$$

An immediate implication is that $c_t^{H,L} = c_t^{L,H}$ for $t = 1, 2$. Notice that the optimal values of k_0 , k_1 , and k_2 are not unique. In particular, k_0 can be any non-negative number (it doesn't even enter the bank maximization problem) while different values for k_1 and k_2 could support the same consumption levels but with different roll-overs. To eliminate this multiplicity, we will focus on the solution that allows for the same amount of roll over in both state (H, L) and (L, H) . More precisely, let (δ^{pi}, k) be optimal and define the following quantities:

$$\begin{aligned} RO_H & = y + k_1 - \omega_H c_1^{s,-s} \\ RO_L & = y - k_1 - \omega_L c_1^{s,-s} \\ \overline{RO} & = \frac{RO_H + RO_L}{2}, \end{aligned}$$

where RO_s denotes the amount of roll over when the shock is s in our region and $-s$ in the other region, and \overline{RO} is the mean roll over. It is then possible to check that if we consider k' instead of k , where we set $k'_t = k_t - \frac{RO_H - RO_L}{2}$ for $t = 1, 2$, then δ^{pi} is still feasible and, therefore, (δ^{pi}, k') is also optimal. However, with k' the amount of roll over in state (H, L) and (L, H) is the same.

Taking into account that period two feasibility constraints are binding, that is $c_t^{H,L} = c_t^{L,H} = c_t^{s,-s}$, and denoting with RO the amount of roll over in case of asymmetric shock, the feasibility constraints for state (H, L) and (L, H) are:

$$\begin{aligned}\omega_H c_1^{s,-s} + RO &= y + k_1 \\ \omega_L c_1^{s,-s} + RO &= y - k_1 \\ (1 - \omega_H) c_2^{s,-s} &= Rx + RO - k_2 \\ (1 - \omega_L) c_2^{s,-s} &= Rx + RO + k_2.\end{aligned}$$

Subtracting the second from the first and the third from the fourth, we have

$$k_1 = (\omega_H - \gamma) c_1^{s,-s}$$

and

$$k_2 = (\omega_H - \gamma) c_2^{s,-s}.$$

■

Proof of Proposition 5. Consider the reduced partial integration problem [RPI]. Optimality requires $x + y = 1$ (no waste of resources) and, as shown in Proposition 2, $\omega_H c_1^{H,H} = y$. The problem then can be simplified as follows:

$$\begin{aligned}\max_{\{y^{pi}, \{c_t^{L,L}, c_t^{s,-s}, c_t^{H,H}\}_{t=1,2}\}} & p [\gamma u(c_1^{s,-s}) + (1 - \gamma) u(c_2^{s,-s})] + \\ & + \frac{1-p}{2} \left[\omega_H u\left(\frac{y}{\omega_H}\right) + (1 - \omega_H) u(c_2^{H,H}) + \omega_L u(c_1^{L,L}) + (1 - \omega_L) u(c_2^{L,L}) \right]\end{aligned}$$

subject to

$$\begin{aligned}\omega_L c_1^{L,L} &\leq y; \\ \gamma c_1^{s,-s} &\leq y; \\ (1 - \omega_H) c_2^{H,H} &\leq R(1 - y); \\ (1 - \omega_L) c_2^{L,L} &\leq R(1 - y) + y - \omega_L c_1^{L,L}; \\ (1 - \gamma) c_2^{s,-s} &\leq R(1 - y) + y - \gamma c_1^{s,-s}; \\ c_t^{s,s} &\geq 0; \quad s = L, H \quad t = 1, 2 \\ c_t^{s,-s} &\geq 0; \quad t = 1, 2 \\ 0 &\leq y \leq 1.\end{aligned}$$

Inada conditions guarantee that $c_t^{s,s} > 0 \forall s$ and $\forall t$, $c_t^{s,-s} > 0 \forall t$, and $y \in (0, 1)$. Let $\mu_t^{s,s}$ be the multiplier associated with the feasibility constraint at time t and state $s = H, L$, and $\mu_t^{s,-s}$ be the multiplier associated with the feasibility constraint at time t and state $s, -s$.

The FOCs are:

$$\frac{1-p}{2}u'\left(\frac{y}{\omega_H}\right) + \mu_1^{L,L} + \mu_1^{s,-s} = \mu_2^{H,H}R + \mu_2^{L,L}(R-1) + \mu_2^{s,-s}(R-1); \quad (13)$$

$$\frac{1-p}{2}u'(c_1^{L,L}) = \mu_1^{L,L} + \mu_2^{L,L}; \quad (14)$$

$$pu'(c_1^{s,-s}) = \mu_1^{s,-s} + \mu_2^{s,-s}; \quad (15)$$

$$pu'(c_2^{s,-s}) = \mu_2^{s,-s}; \quad (16)$$

$$\frac{1-p}{2}u'(c_2^{H,H}) = \mu_2^{H,H}; \quad (17)$$

$$\frac{1-p}{2}u'(c_2^{L,L}) = \mu_2^{L,L}. \quad (18)$$

It follows immediately that $\mu_2^{L,L} > 0$, $\mu_2^{H,H} > 0$, and $\mu_2^{s,-s} > 0$. Then all second period feasibility constraints are binding. From (14) and (18) we have

$$(1-p)u'(c_1^{L,L}) - 2\mu_1^{L,L} = (1-p)u'(c_2^{L,L}) \implies c_1^{L,L} \leq c_2^{L,L},$$

considering (15) and (17) we have

$$pu'(c_1^{s,-s}) - \mu_1^{s,-s} = pu'(c_2^{s,-s}) \implies c_1^{s,-s} \leq c_2^{s,-s}.$$

Then incentive compatibility is satisfied when both shocks are low and when they are negatively correlated. We have four cases to consider:

- $\mu_1^{L,L} > 0$ and $\mu_1^{s,-s} > 0$ (solution without roll over). Then we have $c_1^{L,L} = \frac{y}{\omega_L} > \frac{y}{\omega_H} = c_1^{H,H}$, and this implies that $c_2^{H,H} = \frac{R(1-y)}{1-\omega_H} > \frac{R(1-y)}{1-\omega_L} = c_2^{L,L}$. Moreover, $c_1^{s,-s} = \frac{y}{\gamma} > \frac{y}{\omega_H} = c_1^{H,H}$, which implies that $c_2^{s,-s} = \frac{R(1-y)}{1-\gamma}$. So we have

$$c_1^{H,H} < c_1^{s,-s} < c_1^{L,L} < c_2^{L,L} < c_2^{s,-s} < c_2^{H,H}$$

and y is determined by

$$\frac{1-p}{2}u'(c_1^{H,H}) + \frac{1-p}{2}u'(c_1^{L,L}) + pu'(c_1^{s,-s}) = R\left[\frac{1-p}{2}u'(c_2^{H,H}) + \frac{1-p}{2}u'(c_2^{L,L}) + pu'(c_2^{s,-s})\right]. \quad (19)$$

- $\mu_1^{L,L} = \mu_1^{s,-s} = 0$ (solution with roll over when both shocks are low or asymmetric). In this case we have $c_1^{L,L} = c_2^{L,L} < \frac{y}{\omega_L}$, and $c_1^{s,-s} = c_2^{s,-s} < \frac{y}{\gamma}$. The former implies that

$$c_2^{L,L} = \frac{R(1-y) + y - \omega_L c_1^{L,L}}{1 - \omega_L} \implies c_2^{L,L} = R(1-y) + y.$$

Similarly, $c_2^{s,-s} = R(1-y) + y$. From (13) we have that

$$\frac{1-p}{2}u'(c_1^{H,H}) = \mu_2^{H,H}R + \mu_2^{L,L}(R-1) + \mu_2^{s,-s}(R-1),$$

and since the RHS is greater than $\mu_2^{H,H}$, we have that

$$c_1^{H,H} < c_2^{H,H} = \frac{R(1-y)}{1-\omega_H}.$$

Finally, notice that

$$\begin{aligned} c_2^{H,H} &> c_1^{H,H} \implies \frac{R(1-y)}{1-\omega_H} > \frac{y}{\omega_H} \implies R(1-y)\frac{\omega_H}{1-\omega_H} > y \implies R(1-y)\left[\frac{1}{1-\omega_H} - 1\right] > y \\ &\implies \frac{R(1-y)}{1-\omega_H} > R(1-y) + y \implies c_2^{H,H} > c_2^{L,L} = c_2^{s,-s}. \end{aligned}$$

and that

$$\begin{aligned} c_2^{H,H} &> c_1^{H,H} \implies \frac{R(1-y)}{1-\omega_H} > \frac{y}{\omega_H} \implies R(1-y) > \frac{y}{\omega_H}(1-\omega_H) \implies R(1-y) > \frac{y}{\omega_H} - y \\ &\implies R(1-y) + y > \frac{y}{\omega_H} \implies c_2^{L,L} = c_2^{s,-s} > c_1^{H,H}. \end{aligned}$$

So we have

$$c_1^{H,H} < c_1^{L,L} = c_1^{s,-s} = c_2^{L,L} = c_2^{s,-s} < c_2^{H,H}$$

and y is again determined by (19), which now can be written as

$$\frac{1-p}{2}u'(c_1^{H,H}) = R\frac{1-p}{2}u'(c_2^{H,H}) + (R-1)\left[\frac{1-p}{2}u'(c_2^{L,L}) + pu'(c_2^{s,-s})\right].$$

- $\mu_1^{L,L} = 0$ and $\mu_2^{s,-s} > 0$ (solution with roll over only when the both shocks are low). In this case we have $c_1^{L,L} = c_2^{L,L} = R(1-y) + y < \frac{y}{\omega_L}$, $c_1^{s,-s} = \frac{y}{\gamma}$, and $c_2^{s,-s} = \frac{R(1-y)}{1-\gamma}$. Accordingly, $c_1^{s,-s} > c_1^{H,H}$ and $c_2^{s,-s} < c_2^{H,H}$. From (15) and (16) we have

$$p[u'(c_1^{s,-s}) - u'(c_2^{s,-s})] = \mu_1^{s,-s} > 0 \implies c_1^{s,-s} < c_2^{s,-s}.$$

Notice that

$$\begin{aligned} c_2^{s,-s} &> c_1^{s,-s} \implies \frac{R(1-y)}{1-\gamma} > \frac{y}{\gamma} \implies R(1-y)\frac{\gamma}{1-\gamma} > y \implies R(1-y)\left[\frac{1}{1-\gamma} - 1\right] > y \\ &\implies \frac{R(1-y)}{1-\gamma} > R(1-y) + y \implies c_2^{s,-s} > c_2^{L,L}. \end{aligned}$$

Similarly,

$$c_2^{s,-s} > c_1^{s,-s} \implies \frac{R(1-y)}{1-\gamma} > \frac{y}{\gamma} \implies R(1-y) > \frac{y}{\gamma} - y \implies R(1-y) + y > \frac{y}{\gamma} \implies c_1^{L,L} > c_1^{s,-s}.$$

The solution in this case is

$$c_1^{H,H} < c_1^{s,-s} < c_1^{L,L} = c_2^{L,L} < c_2^{s,-s} < c_2^{H,H}$$

and y is again determined by (19), which now can be written as

$$\frac{1-p}{2}u'(c_1^{H,H}) + pu'(c_1^{s,-s}) = R\left[\frac{1-p}{2}u'(c_2^{H,H}) + pu'(c_2^{s,-s})\right] + (R-1)\frac{1-p}{2}u'(c_2^{L,L}). \quad (20)$$

- $\mu_1^{L,L} > 0$ and $\mu_1^{s,-s} = 0$ (solution with roll over only when the shocks are asymmetric). In this case we have $c_1^{s,-s} = c_2^{s,-s} = R(1-y) + y < \frac{y}{\gamma}$ and $c_1^{L,L} = \frac{y}{\omega_L}$ and $c_2^{L,L} = \frac{R(1-y)}{1-\omega_L}$. Accordingly, $c_1^{L,L} > c_1^{H,H}$ and $c_2^{L,L} < c_2^{H,H}$. From (14) and (18) we have

$$\frac{1-p}{2}[u'(c_1^{L,L}) - u'(c_2^{L,L})] = \mu_1^{L,L} > 0 \implies c_1^{L,L} < c_2^{L,L}.$$

Notice that

$$c_2^{L,L} > c_1^{L,L} \implies \frac{R(1-y)}{1-\omega_L} > \frac{y}{\omega_L} \implies R(1-y) > \frac{y}{\omega_L} - y \implies R(1-y) + y > \frac{y}{\omega_L} \implies c_1^{s,-s} > c_1^{L,L}.$$

This cannot be a solution since $c_1^{L,L} = \frac{y}{\omega_L} > \frac{y}{\gamma} > c_1^{s,-s}$. ■

Proof of Proposition 6. Consider the reduced partial integration problem [RPI]. Let $y(p)$ be the optimal investment in the short asset for a given probability p . Clearly $y(1) = y^*$ and $y(0) = y^a$. Notice that y^* is the solution to:

$$u'(\varphi_1^*(y)) - Ru'(\varphi_2^*(y)) = 0 \quad (21)$$

while y^a solves:

$$u'(\varphi_1^{H(a)}(y)) + u'(\varphi_1^{L(a)}(y)) - R[u'(\varphi_2^{H(a)}(y)) + u'(\varphi_2^{L(a)}(y))] = 0 \quad (22)$$

where φ_1^* and $(\varphi_1^{H(a)}, \varphi_1^{L(a)})$ describe optimal first period consumption in problem [FB] and [A], respectively, as a function of y . Similarly, φ_2^* and $(\varphi_2^{H(a)}, \varphi_2^{L(a)})$ describe optimal second period consumption as a function of y in problem [FB] and [A], respectively. Notice also that $y(p)$ is the unique solution to

$$\begin{aligned} & \frac{1-p}{2} \left\{ u'(\varphi_1^{H,H}(y)) + u'(\varphi_1^{L,L}(y)) - R[u'(\varphi_2^{H,H}(y)) + u'(\varphi_2^{L,L}(y))] \right\} + \\ & + p \left\{ u'(\varphi_1^{s,-s}(y)) - Ru'(\varphi_2^{s,-s}(y)) \right\} = 0, \end{aligned} \quad (23)$$

where φ_1^H , φ_1^s and φ_1^L describe optimal first period consumption in problem [RPI] as a function of y , whereas φ_2^H , φ_2^s and φ_2^L describe optimal second period consumption in the same problem, again as a function of y . All the φ s are continuous functions and it is also possible to check that:

$$\begin{aligned} \varphi_1^*(y) &= \frac{y}{\gamma} & \varphi_2^*(y) &= \frac{1-y}{1-\gamma} R \\ \varphi_1^{H(a)}(y) &= \frac{y}{w_H} & \varphi_2^{H(a)}(y) &= \frac{1-y}{1-w_H} R \\ \varphi_1^{L(a)}(y) &= \min \left\{ \frac{y}{w_L}, (1-y)R + y \right\} & \varphi_2^{L(a)}(y) &= \max \left\{ \frac{1-y}{1-w_L} R, (1-y)R + y \right\} \\ \varphi_1^{H,H}(y) &= \frac{y}{w_H} & \varphi_2^{H,H}(y) &= \frac{1-y}{1-w_H} R \\ \varphi_1^{s,-s}(y) &= \min \left\{ \frac{y}{\gamma}, (1-y)R + y \right\} & \varphi_2^{s,-s}(y) &= \max \left\{ \frac{1-y}{1-\gamma} R, (1-y)R + y \right\} \\ \varphi_1^{L,L}(y) &= \min \left\{ \frac{y}{w_L}, (1-y)R + y \right\} & \varphi_2^{L,L}(y) &= \max \left\{ \frac{1-y}{1-w_L} R, (1-y)R + y \right\}. \end{aligned}$$

Before going through the rest of the proof it will be useful to notice that equations (21), (22) and (23) all have a unique solution. In fact, they represent the first order conditions with respect to y in problems [FB], [A], and [RPI], respectively. All such problems involve the maximization of a strictly quasi-concave objective function on a convex feasible set and, therefore, they admit a unique solution.⁶ Furthermore, given the Inada conditions assumed on the marginal utility u' , the optimal short investment must strictly lie between zero and one and, as a consequence, must satisfy the relevant first order condition. Notice also that for any y to the left of the relevant optimal level, the LHS of (21), (22) and (23) is strictly positive, while for any y to the right of the relevant optimal level the LHS of (21), (22) and (23) is strictly negative.⁷ This argument will be used repeatedly in the remainder of the proof.

To show the first statement of the proposition consider any $p \in (0, 1)$ and assume $y^* < y^a$, so that

$$\begin{aligned} u'(\varphi_1^{H(a)}) + u'(\varphi_1^{L(a)}) - R \left[u'(\varphi_2^{H(a)}) + u'(\varphi_2^{L(a)}) \right] \Big|_{y=y^a} &= 0 \\ \{u'(\varphi_1^*) - Ru'(\varphi_2^*)\} \Big|_{y=y^a} &< 0. \end{aligned}$$

Furthermore, since $\varphi_t^{H(a)}(y) = \varphi_t^{H,H}(y)$ and $\varphi_t^{L(a)}(y) = \varphi_t^{L,L}(y)$ for all y and for both $t = 1, 2$, we also have that

$$u'(\varphi_1^{H,H}) + u'(\varphi_1^{L,L}) - R \left[u'(\varphi_2^{H,H}) + u'(\varphi_2^{L,L}) \right] \Big|_{y=y^a} = 0$$

Note also that for any given y , either $\varphi_t^*(y) = \varphi_t^{s,-s}(y)$ for both $t = 1, 2$, or $\varphi_1^{s,-s}(y) = \varphi_2^{s,-s}(y)$, so that in any case it must be

$$\{u'(\varphi_1^{s,-s}) - Ru'(\varphi_2^{s,-s})\} \Big|_{y=y^a} < 0.$$

This means that the LHS of condition (23) is negative for $y = y^a$ so that its solution $y(p)$ must be smaller than y^a and therefore

$$u'(\varphi_1^{H,H}) + u'(\varphi_1^{L,L}) - R \left[u'(\varphi_2^{H,H}) + u'(\varphi_2^{L,L}) \right] \Big|_{y=y(p)} > 0$$

that, for (23) to hold, in turn implies

$$\{u'(\varphi_1^{s,-s}) - Ru'(\varphi_2^{s,-s})\} \Big|_{y=y(p)} < 0.$$

⁶To be more precise, problem [RPI] allows for multiple solutions in the extreme cases of $p = 1$ and $p = 0$. For example, with $p = 1$, i.e. when the probability of being hit by undiversifiable liquidity shocks is zero, optimal consumption levels $c_t^{H,H}$ and $c_t^{L,L}$ are irrelevant and then indetermined. In any case the optimal short term investment is always unique and satisfies the first order condition.

⁷To see why, notice that the LHS of (21), (22) and (23) diverge to $+\infty$ as y approaches zero and to $-\infty$ as y approaches one. Furthermore, they are continuous functions with exactly one zero, so that they must be strictly positive before it and strictly negative beyond it.

Consider now $p' > p$. It results that:

$$\begin{aligned} & \frac{1-p'}{2} \left\{ u'(\varphi_1^{H,H}) + u'(\varphi_1^{L,L}) - R \left[u'(\varphi_2^{H,H}) + u'(\varphi_2^{L,L}) \right] \right\} + \\ & \quad p' \left\{ u'(\varphi_1^{s,-s}) - Ru'(\varphi_2^{s,-s}) \right\} \Big|_{y=y(p)} < \\ & \frac{1-p}{2} \left\{ u'(\varphi_1^{H,H}) + u'(\varphi_1^{L,L}) - R \left[u'(\varphi_2^{H,H}) + u'(\varphi_2^{L,L}) \right] \right\} + \\ & \quad p \left\{ u'(\varphi_1^{s,-s}) - Ru'(\varphi_2^{s,-s}) \right\} \Big|_{y=y(p)} = 0 \end{aligned}$$

which immediately implies $y(p') < y(p)$.

As for the second statement of the proposition consider again any $p \in (0, 1)$ and assume $y^* > y^a$, so that

$$\begin{aligned} & u'(\varphi_1^{H(a)}) + u'(\varphi_1^{L(a)}) - R \left[u'(\varphi_2^{H(a)}) + u'(\varphi_2^{L(a)}) \right] \Big|_{y=y^a} = 0 \\ & \quad \left\{ u'(\varphi_1^*) - Ru'(\varphi_2^*) \right\} \Big|_{y=y^a} > 0. \end{aligned}$$

Again, since for both $t = 1, 2$ and for all y we have that $\varphi_t^{H(a)}(y) = \varphi_t^{H,H}(y)$, $\varphi_t^{L(a)}(y) = \varphi_t^{L,L}(y)$ and $\varphi_1^{s,-s}(y) \leq \varphi_1^*$, $\varphi_2^{s,-s}(y) \geq \varphi_2^*(y)$, it results

$$\begin{aligned} & u'(\varphi_1^{H,H}) + u'(\varphi_1^{L,L}) - R \left[u'(\varphi_2^{H,H}) + u'(\varphi_2^{L,L}) \right] \Big|_{y=y^a} = 0 \\ & \quad \left\{ u'(\varphi_1^{s,-s}) - Ru'(\varphi_2^{s,-s}) \right\} \Big|_{y=y^a} > 0. \end{aligned}$$

This means that the LHS of condition (23) is positive when evaluated at $y = y^a$ so that its solution $y(p)$ must be larger than y^a and, therefore, it must be that

$$u'(\varphi_1^{H,H}) + u'(\varphi_1^{L,L}) - R \left[u'(\varphi_2^{H,H}) + u'(\varphi_2^{L,L}) \right] \Big|_{y=y(p)} < 0$$

that, for (23) to hold, in turn implies

$$\left\{ u'(\varphi_1^{s,-s}) - Ru'(\varphi_2^{s,-s}) \right\} \Big|_{y=y(p)} > 0.$$

Consider now $p' > p$. It results that

$$\begin{aligned} & \frac{1-p'}{2} \left\{ u'(\varphi_1^{H,H}) + u'(\varphi_1^{L,L}) - R \left[u'(\varphi_2^{H,H}) + u'(\varphi_2^{L,L}) \right] \right\} + \\ & \quad p' \left\{ u'(\varphi_1^{s,-s}) - Ru'(\varphi_2^{s,-s}) \right\} \Big|_{y=y(p)} > \\ & \frac{1-p}{2} \left\{ u'(\varphi_1^{H,H}) + u'(\varphi_1^{L,L}) - R \left[u'(\varphi_2^{H,H}) + u'(\varphi_2^{L,L}) \right] \right\} + \\ & \quad p \left\{ u'(\varphi_1^{s,-s}) - Ru'(\varphi_2^{s,-s}) \right\} \Big|_{y=y(p)} = 0 \end{aligned}$$

that immediately implies $y(p') > y(p)$.

Conversely, if $y(p)$ is decreasing in p , then $y^* = y(0) < y(1) = y^a$. While, if $y(p)$ is increasing, then $y^* = y(0) > y(1) = y^a$. ■

Proof of Proposition 7. In state (H, H) there is never roll over and the cost of liquidity is

$$CL^{H,H} = \frac{c_2^{H,H}}{c_1^{H,H}} = \frac{\omega_H}{1 - \omega_H} \frac{(1 - y^{pi})}{y^{pi}} R.$$

Given proposition 6, if $y^* < y^a$, then y^{pi} is strictly decreasing in p and, as a consequence, $CL^{H,H}$ is strictly increasing in p in this case. Similarly, under the assumption that $y^* < y^a$, in the range of p in which there is no roll over in state (s_1, s_2) , the cost of liquidity CL^{s_1, s_2} is strictly increasing in p , while in the range of p in which there is roll over in state (s_1, s_2) , the cost of liquidity CL^{s_1, s_2} is constant and equal to one. To complete the argument just notice that for the state $(s_1, s_2) \in \{(s, -s), (L, L)\}$, there is a threshold $\bar{p}(s_1, s_2)$ such that if $\bar{p} < p(s_1, s_2)$, there is no roll over in state (s_1, s_2) , while if $p > \bar{p}(s_1, s_2)$, there is roll over in state (s_1, s_2) . The threshold $\bar{p}(s, -s)$ is zero if $\frac{y^{pi}(0)}{\gamma} < [1 - y^{pi}(0)] R + y^{pi}(0)$ and it is one if $\frac{y^{pi}(1)}{\gamma} > [1 - y^{pi}(1)] R + y^{pi}(1)$, otherwise it is implicitly defined by

$$\frac{y^{pi}(p)}{\gamma} = [1 - y^{pi}(p)] R + y^{pi}(p).$$

Similarly, the threshold $\bar{p}(L, L)$ is zero if $\frac{y^{pi}(0)}{\omega_L} < [1 - y^{pi}(0)] R + y^{pi}(0)$ and it is one if $\frac{y^{pi}(1)}{\omega_L} > [1 - y^{pi}(1)] R + y^{pi}(1)$, otherwise it is implicitly defined by

$$\frac{y^{pi}(p)}{\omega_L} = [1 - y^{pi}(p)] R + y^{pi}(p).$$

Clearly, $\bar{p}(s, -s) \leq \bar{p}(L, L)$. ■

Proof of Proposition 8. For a given p , consider the cost of liquidity in states (L, L) and $(s, -s)$. In state (L, L) it is either $CL^{L,L} = 1$ if roll over is optimal (in this case $c_2^{L,L} = c_1^{L,L}$) or

$$CL^{L,L} = \frac{c_2^{L,L}}{c_1^{L,L}} = \frac{\omega_L}{1 - \omega_L} \frac{(1 - y^{pi})}{y^{pi}} R$$

if no roll over is optimal. In both cases $CL^* > CL^{L,L}$. In fact $CL^* = \frac{\gamma}{1 - \gamma} \frac{(1 - y^*)}{y^*} R > 1$, and $\omega_L < \gamma$, while the assumption $y^a > y^*$, together with proposition 6, implies $y^{pi} > y^*$. Similarly, the cost of liquidity in state $(s, -s)$ is either $CL^{s,-s} = 1$ if roll over is optimal (in this case $c_2^{s,-s} = c_1^{s,-s}$) or

$$CL^{s,-s} = \frac{c_2^{s,-s}}{c_1^{s,-s}} = \frac{\gamma}{1 - \gamma} \frac{(1 - y^{pi})}{y^{pi}} R$$

if no roll over is optimal. In both cases $CL^* > CL^{s,-s}$. Finally, if roll over is optimal in state $(s, -s)$ then it is also optimal in state (L, L) but not vice versa, so that we have $CL^{L,L} \leq CL^{s,-s}$.

For a given p , consider now the cost of liquidity in state (H, H) . We want to show that $CL^* < CL^{H,H}$. This last inequality is implied by $c_1^{H,H} < c_1^*$ and $c_2^* < c_2^{H,H}$, that in

turn, given proposition 6 and the assumption $y^a > y^*$, is implied by $c_1^H < c_1^*$ and $c_2^* < c_2^H$ (where c_t^H is the autarky level of consumption at time t in a region hit by the high liquidity shock). The last two inequalities simply state the intuitive result that when a region is hit by the high liquidity shock, early consumers can consume less in autarky than in perfect integration, while late consumers consume more in autarky than in perfect integration. To show this result let's proceed by contradiction. We distinguish three cases:

1) If $c_1^H \geq c_1^*$ and $c_2^* \geq c_2^H$, since $c_1^H < c_1^L \leq c_2^L < c_2^H$, we obtain

$$2u'(c_1^*) > u'(c_1^H) + u'(c_1^L) = R[u'(c_2^H) + u'(c_2^L)] > 2Ru'(c_2^*)$$

which contradicts the fact that (c_1^*, c_2^*) are first best consumption levels.

2) If $c_1^H \geq c_1^*$ and $c_2^* < c_2^H$, then ... contradiction (probably some feasibility condition is violated). 3) Finally, if $c_1^H < c_1^*$ and $c_2^* \geq c_2^H$, then ... contradiction (probably some feasibility condition is violated). TBF ■

Proof of Proposition 9. If roll over is optimal in both states (L, L) and $(s, -s)$, then $\bar{c}_1^{s,-s} = \bar{c}_1^{L,L} = R(1 - y^{pi}) + y^{pi}$ and $\bar{c}_1^{H,H} = \frac{y^{pi}}{\omega_H}$. It is then possible to verify that

$$0 < \frac{\bar{c}_1^{L,L} - \bar{c}_1^{H,H}}{2} = \bar{c}_1^{s,-s} - \bar{c}_1^{s,s} = \frac{\omega_H R - y(p)(1 + \omega_H(1 + R))}{2\omega_H}$$

so that the expression of the first period consumption variance reduces to

$$Var^{pi}(p) = (1 - p^2) \left[\frac{\omega_H R - y(p)(1 + \omega_H(1 + R))}{2\omega_H} \right]^2.$$

The statement in the proposition is equivalent to say that the right-sided derivative of the consumption variance in $p = 0$ is strictly positive. If we define the positive quantity $\rho = \frac{1 + \omega_H(1 + R)}{2\omega_H}$ and the positive and bounded function

$$\Phi(p) = \left[\frac{\omega_H R - y(p)(1 + \omega_H(1 + R))}{2\omega_H} \right],$$

the first derivative of the consumption variance at any $p \in (0, 1)$ is given by

$$\frac{dVar^{pi}(p)}{dp} = -2p\Phi^2(p) - 2\rho(1 - p^2)\Phi(p)y'(p).$$

The right-sided derivative at $p = 0$ is:

$$\frac{dVar^{pi}(0)}{dp} = \lim_{p \rightarrow 0^+} \frac{dVar^{pi}(p)}{dp} = -2\rho\Phi(0)y'(0),$$

where $y'(0) = \lim_{p \rightarrow 0^+} y'(p)$ denotes the right-sided derivative of $y(p)$ at $p = 0$.

Notice that if $y^* \geq y^a$ then $y'(p) \geq 0$ and $\frac{dVar^{pi}(p)}{dp} \leq 0$ for any $p \in (0, 1)$ so that $\frac{dVar^{pi}(0)}{dp} \leq 0$. However, if $y^* < y^a$ then $y'(p) < 0$ for any $p \in (0, 1)$ and, therefore, $\frac{dVar^{pi}(0)}{dp} \geq 0$. For the

last inequality to be strict it is necessary that $y'(0)$ be strictly negative. This is surely true if rolling over is optimal in both states (L, L) and $(s, -s)$. In fact, in this case, differentiating with respect to $p \in (0, 1)$ the first order condition (23), it is possible to verify that

$$y'(p) = \frac{u' \left(\frac{y(p)}{\omega_H} \right) - Ru' \left(\frac{1-y(p)}{1-\omega_H} R \right) - (1-R)u' (R(1-y(p)) + y(p))}{\frac{1-p}{\omega_H} u'' \left(\frac{y(p)}{\omega_H} \right) + \frac{1-p}{1-\omega_H} R^2 u'' \left(\frac{1-y(p)}{1-\omega_H} R \right) + (1-R)^2 (1+p) u'' (R(1-y(p)) + y(p))}.$$

Notice that the denominator of the above expression is strictly negative for any $p \in (0, 1)$ as well as its limit for $p \rightarrow 0^+$. As a consequence, the sign of $y'(0)$ is given by

$$- \left[u' \left(\frac{y^a}{\omega_H} \right) - Ru' \left(\frac{1-y^a}{1-\omega_H} R \right) - (1-R)u' (R(1-y^a) + y^a) \right]. \quad (24)$$

To complete the proof, notice that the first order condition (22) in case of roll over reduces to:

$$u' \left(\frac{y^a}{\omega_H} \right) - Ru' \left(\frac{1-y^a}{1-\omega_H} R \right) + (1-R)u' (R(1-y^a) + y^a) = 0$$

which immediately implies that (24) is strictly negative. ■

Proof of Lemma 1. Consider a random variable that, as the first period consumption in partial integration, can only assume three possible values $x_1 < x_2 \leq x_3$ with probabilities $\frac{1-p}{2}$, p and $\frac{1-p}{2}$. The sign of its skewness is equal to the sign of its third moment M_3 that, after some algebra can be written as follows

$$M_3 = \frac{p(1-p)(x_1 + x_3 - 2x_2)}{8} [(2p-1)(x_1 + x_3 - 2x_2)^2 + 3(x_3 - x_1)^2].$$

We want to show that such quantity has the sign of $[(x_3 - x_2) - (x_2 - x_1)]$. Notice that $[(x_3 - x_2) - (x_2 - x_1)] = (x_1 + x_3 - 2x_2)$ and therefore the lemma immediately follows from

$$[(2p-1)(x_1 + x_3 - 2x_2)^2 + 3(x_3 - x_1)^2] > 0.$$

This last quantity is increasing in p so that it suffices to show that it is positive for $p = 0$, that is

$$[3(x_3 - x_1)^2 - (x_1 + x_3 - 2x_2)^2] > 0.$$

To this end, notice that from $x_1 < x_2 \leq x_3$ it follows that $(x_3 - 2x_1) > (x_3 - 2x_2) \iff (x_3 - x_1) > (x_1 + x_3 - 2x_2)$ that clearly establishes the lemma. ■

Proof of Proposition 10. Roll over is never optimal if

$$\frac{y^{pi}}{w_L} \leq (1 - y^{pi})R + y^{pi},$$

which is equivalent to the condition

$$y^{pi} \leq \frac{\omega_L R}{\omega_L R + 1 - \omega_L} \equiv \bar{y}.$$

In this case, consumption skewness is surely positive. On the other hand, roll over is optimal both in state (L, L) and $(s, -s)$ if

$$\frac{y^{pi}}{\gamma} > (1 - y^{pi})R + y^{pi},$$

which is equivalent to the condition

$$y^{pi} > \frac{\gamma R}{\gamma R + 1 - \gamma} \equiv \underline{y}.$$

In this second case consumption skewness is surely negative. If $y^{pi} \in (\underline{y}, \bar{y}]$ roll over is optimal in state (L, L) but not in state $(s, -s)$ so that

$$\bar{c}_1^{H,H} = \frac{y^{pi}}{\omega_H}, \bar{c}_1^{s,-s} = \frac{y^{pi}}{\gamma}$$

and

$$\bar{c}_1^{L,L} = (1 - y^{pi})R + y^{pi}.$$

The sign of the consumption skewness is then given by

$$\left(\bar{c}_1^{L,L} - \bar{c}_1^{s,-s} \right) - \left(\bar{c}_1^{s,-s} - \bar{c}_1^{H,H} \right).$$

Substituting consumption levels with their expression as functions of y^{pi} , it is possible to verify that the skewness has the sign of $\tilde{y} - y^{pi}$, where

$$\tilde{y} = \frac{\gamma \omega_H R}{\gamma \omega_H R + 2\omega_H - \gamma(1 + \omega_H)} \in (\underline{y}, \bar{y}).$$

Let $y(p) = y^{pi}$ the optimal short term investment in partial integration as a function of p . We can now show the three statements in Proposition 8:

1. If $\tilde{y} \leq y^* < y^a$, since $y(p) \geq y^*$ for each p , then consumption skewness is negative for each p .
2. If $y^* < \tilde{y} < y^a$, since $y(p)$ is a continuous and strictly decreasing function of p describing the range $[y^*, y^a]$ for $p \in [0, 1]$, there must be $\tilde{p} \in (0, 1)$ such that $y(\tilde{p}) = \tilde{y}$, $y(p)|_{p < \tilde{p}} > \tilde{y}$ and $y(p)|_{p > \tilde{p}} < \tilde{y}$. Given the definition of \tilde{y} , the probability threshold \tilde{p} has therefore the property claimed in the proposition.
3. If $y^* < y^a \leq \tilde{y}$, since $y(p) \leq y^a$ for each p , then consumption skewness is positive for each p . ■

References

- [1] Acharya, V., D. Gromb and T. Yorulmazer (2008), ‘Imperfect Competition in the Inter-Bank Market for Liquidity as a Rationale for Central Banking’, mimeo, London Business School.
- [2] Allen, F., E. Carletti and D. Gale (2008), ‘Interbank Market Liquidity and Central Bank Intervention.’, mimeo, Wharton School.
- [3] Allen, F., and D. Gale (2000), ‘Financial Contagion’, *Journal of Political Economy*, **108**: 1-33.
- [4] Allen, F., and D. Gale (2004), ‘Financial Intermediaries and Markets’, *Econometrica*, **72**: 1023-1061.
- [5] Barro, R. J. (2006), ‘Rare Disasters and Asset Markets in the Twentieth Century’, *Quarterly Journal of Economics*, **121**: 823-866.
- [6] Bhattacharya, S. and D. Gale (1987), ‘Preference Shocks, Liquidity and Central Bank Policy’, in W. Barnett and K. Singleton (eds), ‘*New Approaches to Monetary Economics*’, Cambridge University Press, Cambridge, pp. 69-88.
- [7] Brusco, S., and F. Castiglionesi (2007), ‘Liquidity Coinsurance, Moral Hazard, and Financial Contagion’, *Journal of Finance*, **62** (5): 2275 - 2302.
- [8] Castiglionesi, F. and W. Wagner (2008), ‘The Optimality of Interbank Liquidity Insurance’, Manuscript, Tilburg University.
- [9] Diamond, D., and P. Dybvig (1983), ‘Bunk Runs, Deposit Insurance and Liquidity’, *Journal of Political Economy*, **91**: 401-419.
- [10] Freixas, X., and C. Holthausen (2005), ‘Market Interbank Integration under Asymmetric Information’, *Review Financial Studies*, **18** (2): 459-490.
- [11] Freixas, X., B. Parigi and J. C. Rochet (2000), ‘Systemic Risk, Interbank Relations and Liquidity Provision by the Central Bank’, *Journal of Money, Credit and Banking*, **32**: 611-638.
- [12] Gourio, F. (2008), ‘Disaster and Recoveries’, *American Economic Review P&P*, **98** (2):68-73.
- [13] Rietz, T. (1988), ‘The Equity Premium: a Solution’, *Journal of Monetary Economics*, **22**: 117-131.
- [14] Rochet, J. C. and J. Tirole (1996), ‘Interbank Lending and Systemic Risk’, *Journal of Money, Credit and Banking*, **28**: 733-762.

- [15] Wagner, W. (2008), ‘The Homogenization of the Financial System and Liquidity Crisis’, *Journal of Financial Intermediation*, **17**: 330-356.

Figure1

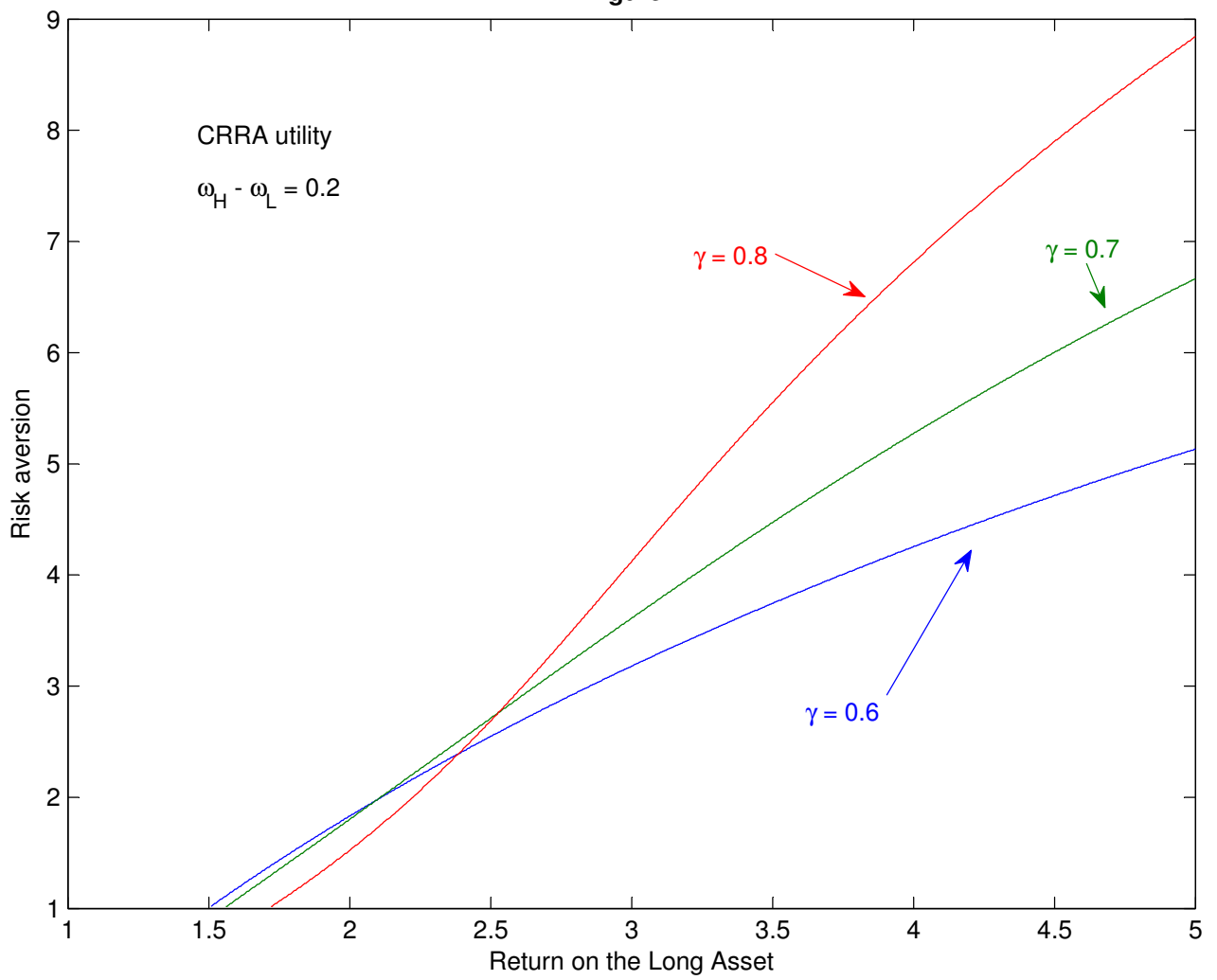


Figure 2

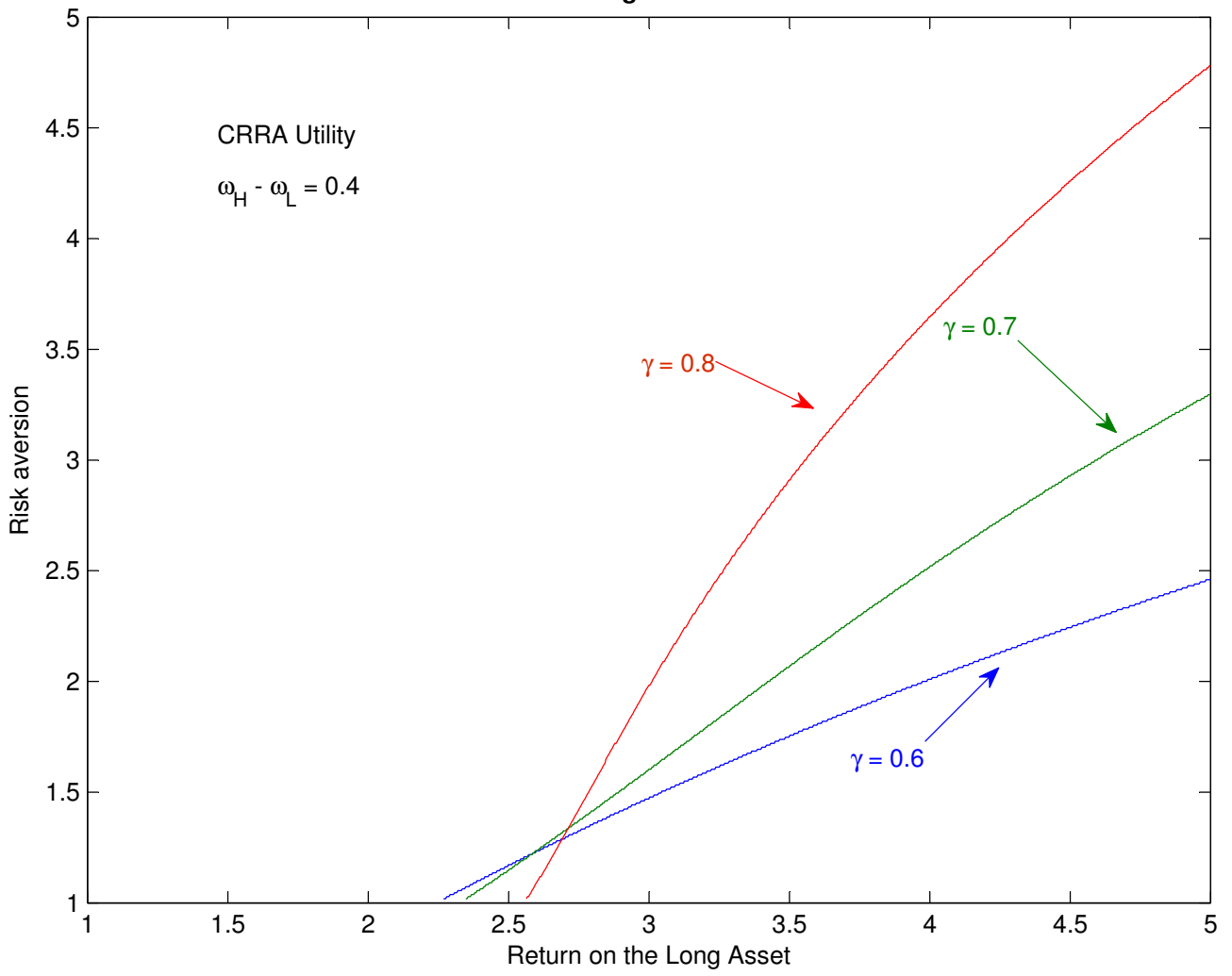


Figure 3: Cost of liquidity

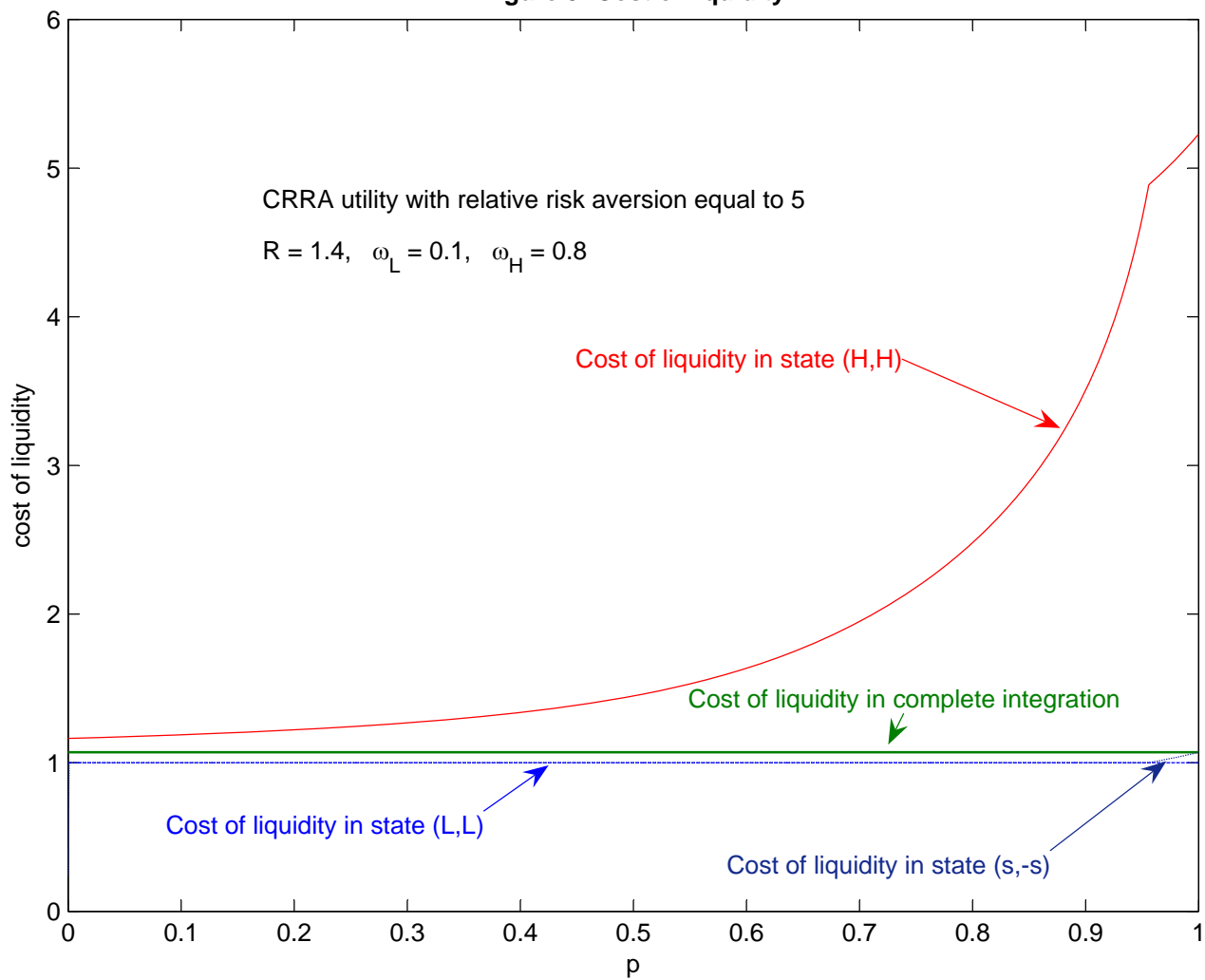


Figure 4: Cost of liquidity

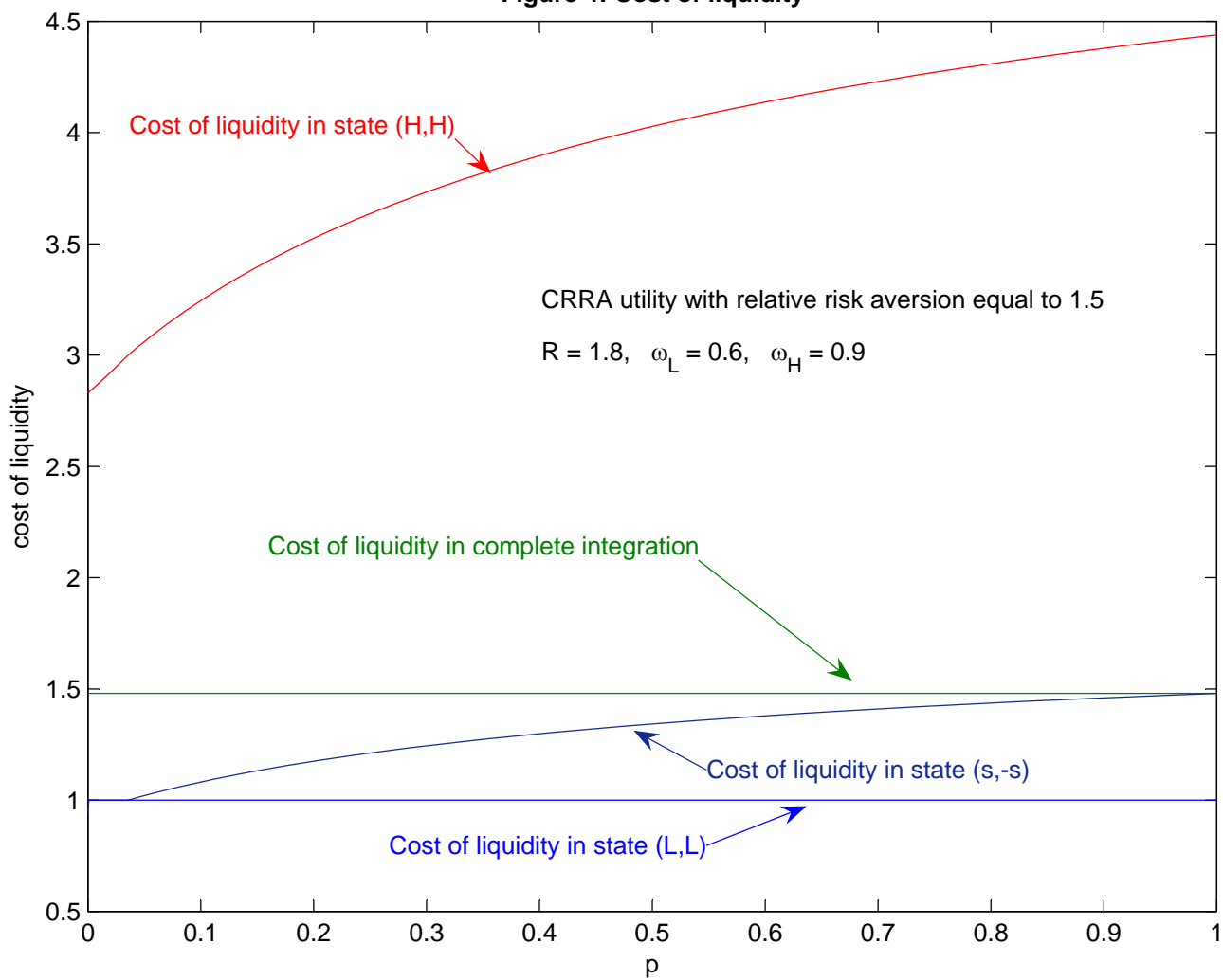


Figure 5: First period consumption volatility

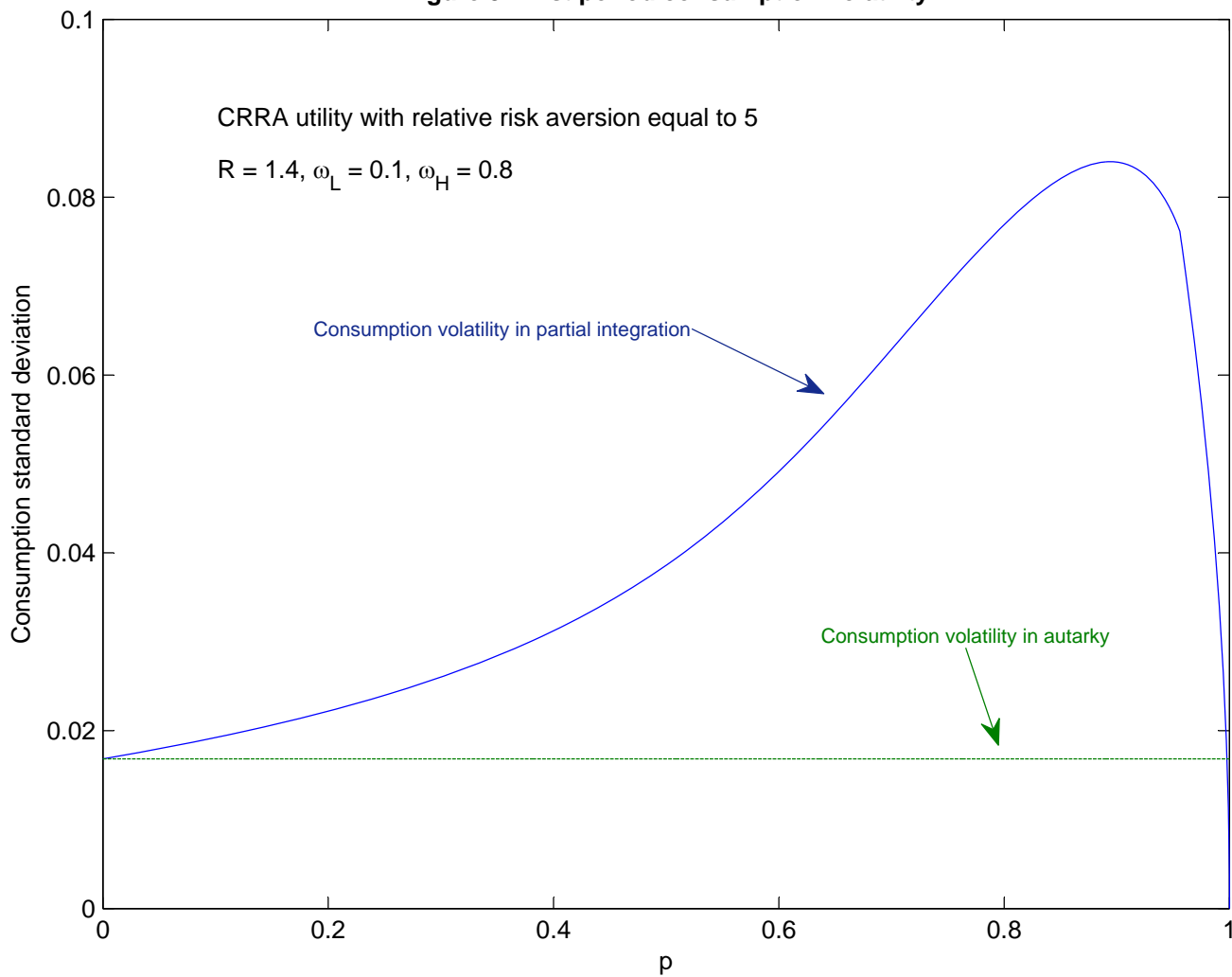


Figure 6: First period consumption volatility

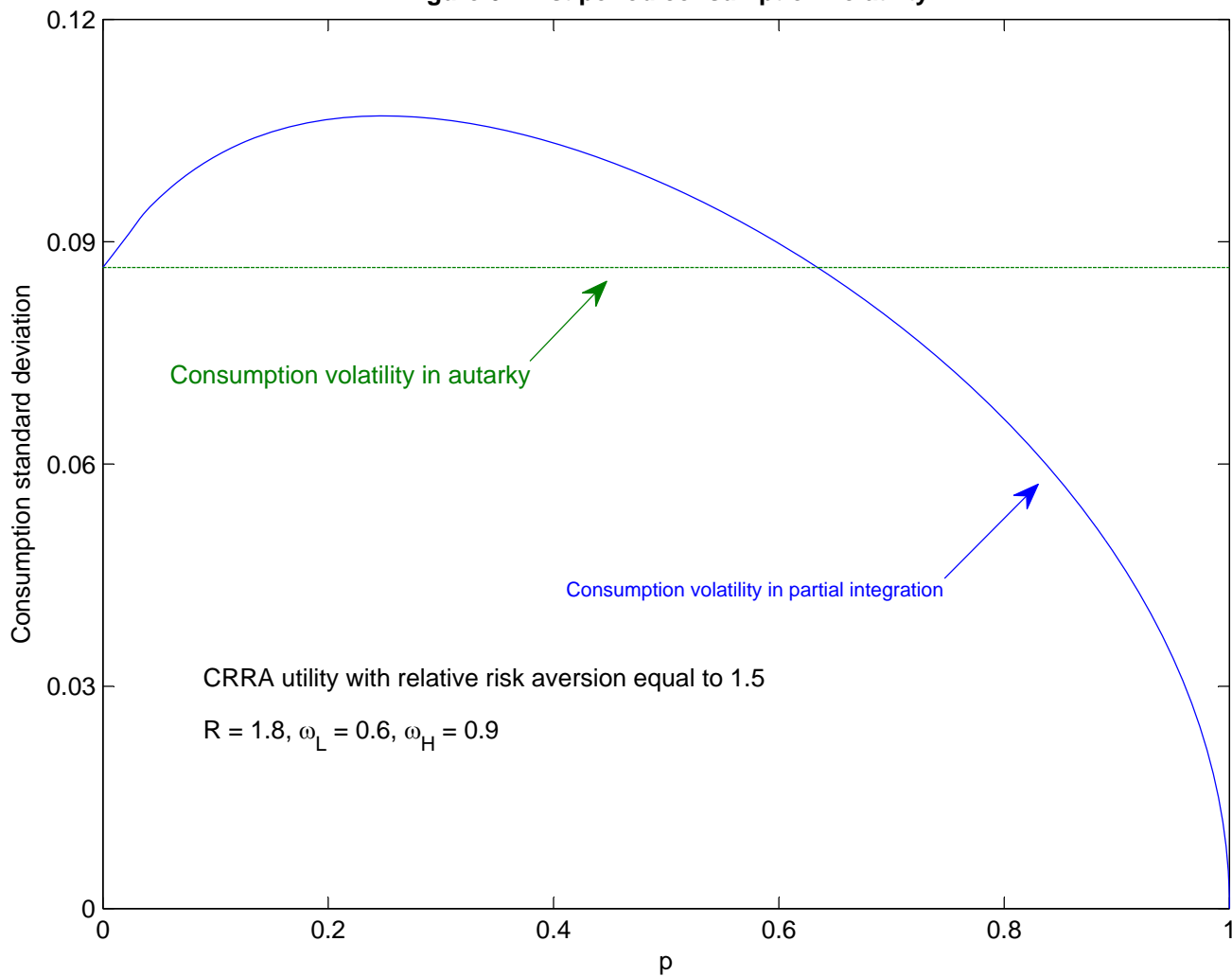


Table 1: Different cases in proposition 8

Long Return	$\omega_L=0.35; \omega_H=0.65; \gamma=0.5$		$\omega_L=0.1; \omega_H=0.9; \gamma=0.5$	
	Switching threshold from case 3 to case 2	Switching threshold from case 2 to case 1	Switching threshold from case 3 to case 2	Switching threshold from case 2 to case 1
1.1	0.19	0.25	0.07	0.15
1.2	0.36	0.48	0.13	0.28
1.4	0.68	0.88	0.25	0.52
1.8	1.19	1.56	0.46	0.92
2.2	1.61	2.07	0.64	1.24
2.6	1.97	2.52	0.81	1.50
3.0	2.28	2.89	0.95	1.72
4.0	2.89	3.65	1.27	2.18
5.0	3.36	4.24	1.52	2.53

Table 2: Consumption Standard Deviation and Skewness

$R = 1.4; \omega_L=0.35; \omega_H=0.65; \gamma=0.5$

p	Risk aversion = 4 (case 1)		Risk aversion = 0.75 (case 2)		Risk aversion = 0.5 (case 3)	
	Std	Skew	Std	Skew	Std	Skew
0	0.036	0	0.204	0	0.275	0
0.1	0.040	-0.201	0.200	-0.032	0.262	0.091
0.2	0.045	-0.408	0.194	-0.040	0.248	0.192
0.3	0.050	-0.629	0.186	-0.028	0.233	0.306
0.4	0.058	-0.873	0.175	0.002	0.217	0.438
0.5	0.066	-1.155	0.162	0.053	0.199	0.596
0.6	0.077	-1.500	0.147	0.128	0.178	0.794
0.7	0.089	-1.960	0.129	0.237	0.155	1.062
0.8	0.080	-2.653	0.107	0.411	0.127	1.477
0.9	0.058	-4.023	0.076	0.763	0.090	2.334
1	0	0	0	0	0	0

Table 3: Consumption Standard Deviation and Skewness

$$R = 2.6; \omega_L = 0.1; \omega_H = 0.9; \gamma = 0.5$$

p	Risk aversion = 4 (case 1)		Risk aversion = 1.2 (case 2)		Risk aversion = 0.5 (case 3)	
	Std	Skew	Std	Skew	Std	Skew
0	0.073	0	0.312	0	0.885	0
0.1	0.095	-0.201	0.354	-0.177	0.866	0.179
0.2	0.118	-0.408	0.368	-0.277	0.841	0.369
0.3	0.142	-0.629	0.368	-0.313	0.809	0.577
0.4	0.165	-0.873	0.359	-0.295	0.770	0.809
0.5	0.185	-1.155	0.343	-0.226	0.721	1.080
0.6	0.203	-1.500	0.319	-0.103	0.661	1.413
0.7	0.215	-1.960	0.286	0.089	0.587	1.860
0.8	0.191	-2.650	0.241	0.393	0.490	2.543
0.9	0.137	-3.941	0.176	0.988	0.354	3.956
1	0	0	0	0	0	0