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Limitations on the Effectiveness of Forward Guidance at the Zero Lower Bound

*Andrew Levin, David Lopez-Salido, Edward Nelson and Tack Yun

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Andrew Levin, David López-Salido, Edward Nelson, and Tack Yun
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Abstract

The recent literature has shown that forward guidance regarding the future path of interest rates can be very effective in preserving macroeconomic stability in the face of an contractionary demand shock, even when the near-term path of the policy rate is constrained by the zero lower bound; moreover, that literature apparently leaves little scope for any further improvements in stabilization performance via fiscal stimulus or nontraditional monetary policies. In this paper, we characterize optimal policy under commitment in a prototypical New Keynesian model and examine whether those conclusions are sensitive to the specification of the shock process and to the interest elasticity of aggregate demand. Although forward guidance is effective in offsetting natural rate shocks of moderate size and persistence, we find that the macroeconomic outcomes are much less appealing for larger and more persistent shocks, especially when the interest elasticity parameter is calibrated to match empirical estimates rather than being set to a much smaller value. Thus, while forward guidance could be sufficient for mitigating the effects of a "Great Moderation"-style shock, a combination of forward guidance *and* other monetary policy measures–such as large-scale asset purchases–might well be called for in responding to a "Great Recession"-style shock.

JEL classification: E32; E43; E52.

Keywords: Optimal Policy under Commitment; Zero Lower Bound; Expectations Management

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Authors' email: andrew.levin@frb.gov; david.j.lopez-salido@frb.gov; edward.nelson@frb.gov; and tack.yun@frb.gov. We thank Gary Anderson, Bill English, Jinill Kim, Brian Madigan, and Elmar Mertens for useful discussions, and Kathleen Easterbrook for excellent research assistance. The views expressed in this paper are solely the responsibility of the authors, and should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or of any other person associated with the Federal Reserve System.

1 Introduction

The worldwide policy response to the financial turmoil and economic downturn of the past two years has consisted of a range of elements. On the monetary side, the traditional lender-of-last-resort function of the monetary authority has been deployed vigorously by the world's major central banks.[1] It is, however, monetary policy actions undertaken with the specific aim of providing macroeconomic stabilization, rather than principally aimed at aiding day-to-day financial market functioning, that we focus upon in this paper. We consider two types of policy measures: *forward guidance* regarding the anticipated future path of short-term nominal interest rates; and *nontraditional policies*, such as large-scale asset purchases (LSAPs) that are intended to provide additional stimulus to the macroeconomy even when the near-term path of the policy rate is constrained by the zero lower bound.

Recent studies of optimal monetary policy at the ZLB have mainly focused on the benefits of forward guidance, with only modest consideration of nontraditional policy measures.[2] This approach suggests that policymakers should announce that the policy rate will be kept low during the initial stages of economic recovery. Such a commitment can provide stimulus to the economy by lowering expected future real interest rates. The expectations of lower real rates arise from two sources in a New Keynesian model: from expectations of low future nominal interest rates, and from higher expected rates of future inflation (Woodford, 1999, p. 302). For example, Eggertsson and Woodford (2003), Nakov (2008), and Walsh (2009) have emphasized the extent to which forward guidance is very effective in stabilizing the output gap and inflation–that is, this policy avoids deflation in the near term while producing only mildly elevated rates of inflation in subsequent periods.

In contrast, nontraditional monetary actions have featured prominently in the policy responses of several major central banks during the first three quarters of 2009. For example, the Bank of England and the Federal Reserve have each engaged in LSAPs, though with different emphases that partly reflect differences in the financial market structures of these two

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[1] See Madigan (2009) for a discussion of recent Federal Reserve lender-of-last-resort activities, and their relation to classic principles of central banking.

[2] Several early studies focused on the use of simple feedback rules for conducting monetary policy at/near the ZLB; see Wolman (1998), Reifschneider and Williams (1999), and Coenen and Wieland (2003). For analysis of optimal policy under commitment at the ZLB, see Jung et al. (2001, 2005), Adam and Billi (2003, 2006), Eggertsson and Woodford (2003), and Nakov (2008).

economies; see Bean (2009, p. 22).[3] The Bank of England's purchases have concentrated on longer-term U.K. government securities (gilts); the Federal Reserve's LSAP program began with, and has had as its largest component, purchases of mortgage-related marketable assets (agency debt and mortgage-backed securities), but has also included a $300 billion program of purchases of long-term U.S. Treasury securities. Nonetheless, the prevailing consensus–as elegantly summarized by Walsh (2009)–casts doubt on the value of nontraditional policies. Simply put, the key question is: Why not simply rely on forward guidance, since it could deliver all the stimulus required at the ZLB, and at little cost in terms of inflation in later periods?

Our analysis in this paper addresses this question by considering optimal policy under commitment in a prototypical New Keynesian model and examining the extent to which the stabilization performance of forward guidance depends on the specification of the shock process and on the interest elasticity of aggregate demand. Throughout this analysis, we abstract from potential issues of imperfect credibility and focus on the case–as in nearly all of the existing literature–where the central bank has a perfect commitment technology.

Although forward guidance is quite effective in offsetting a natural rate shock of moderate size and persistence, we show that the macroeconomic outcomes are much less satisfactory for a larger and more persistent shock, especially when the interest elasticity parameter is calibrated to match empirical estimates rather than being set to a much smaller value. Even though the outcomes under the optimal commitment policy are preferable to those under discretion, the economy still experiences a steep initial decline in output and a marked swing in the inflation rate. Thus, while forward guidance may be sufficient to mitigate the effects of a "Great Moderation"-style shock, a combination of forward guidance *and* other policy measures–such as fiscal stimulus or nontraditional monetary policies–might well be called for in responding to a "Great Recession"-style shock.

Our analysis also points toward reconsidering other aspects of the prevailing consensus regarding monetary policy strategies at the ZLB. First, we find that the constant price level targeting rule proposed by Eggertsson and Woodford (2003) does *not* necessarily provide

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[3] Because it includes central bank purchases of government assets, our conception of nontraditional monetary policies is broader than that of Gertler and Karadi (2009), who refer to "unconventional monetary policy."

a close approximation to the optimal commitment policy. When the economy is hit by a large and persistent natural rate shock, the optimal policy involves a commitment to a persistently elevated inflation rate–roughly similar to the policy prescription advocated by Krugman (1998)–that pushes down the *ex ante* real interest rate and thereby dampens the initial impact of the shock. In contrast, the constant price level targeting rule generates an initial phase of deflation–with a much steeper drop in output than under the optimal policy– and a subsequent phase of positive inflation that eventually brings the price level back to target.

Second, the optimal policy path does *not* necessarily involve a sharp tightening once the policy rate moves up from the ZLB. As noted by Walsh (2009), this feature is characteristic of the optimal policy path obtained by Eggertsson and Woodford (2003), who focused on two-state Markov shocks, and by Adam and Billi (2003, 2006) and Nakov (2008), who focused on relatively transitory autoregressive (AR) shocks. In contrast, our analysis shows that the pace of policy tightening may be quite gradual when the natural rate shock follows a more persistent AR process.

The remainder of this paper is organized as follows. Section 2 highlight some key features of the recent economic downturn and monetary policy responses in six major industrial economies. Section 3 describes our methodology for obtaining the perfect-foresight representation of optimal policy under commitment in the presence of the ZLB. Section 4 quantifies the limitations of forward guidance, while Section 5 presents further sensitivity analysis of these results. Section 6 extends the analysis to the case of uncertainty about the pace of recovery. Section 7 offers some brief concluding remarks.

2 The Recent Experiences of Six Industrial Economies

In this section we outline some of the features of the recent economic downturn and the monetary policy response.

2.1 The Magnitude and Persistence of the Economic Downturn

Table 1 depicts the OECD estimates and forecasts for the output gap in six economies for 2008-2010, as given in the June 2009 issue of its *Economic Outlook*. The table brings out the scale and speed of the deterioration in economic activity. No country had output more than 0.5 percent below potential in 2008.[4] No country is projected to have an output gap *less* negative than minus 4.7 percent in 2009. In no economy is the deterioration in the output gap from 2008 to 2009 less than four percentage points, and in two economies (Japan and Sweden) it is greater than seven percentage points. Moreover, the OECD's projections as of June 2009 suggest that no economy will experience an improvement in the output gap in 2010.

Table 1. Recent and Projected Output Gaps
(*OECD Economic Outlook*, June 2009

	2008	2009	2010	Std dev (91-07)
United States	-0.5	-4.9	-5.4	1.3
Euro Area	0.4	-5.5	-6.0	1.2
Japan	1.3	-6.1	-6.1	2.0
United Kingdom	0.4	-5.4	-6.4	0.9
Canada	-0.4	-4.7	-5.4	2.1
Sweden	-0.1	-7.7	-8.7	2.1

Some longer-term perspective on this change in circumstances is provided by the final column of Table 1, which gives the standard deviation of the output gap for each economy using quarterly OECD gap estimates for the period 1991 to 2007. For each economy, the deterioration in the output gap from 2008 to 2009 is large relative to the historical standard deviation. The shift amounts to two standard deviations for Canada; more than three

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[4] These numbers refer to annual averages, so the deterioration late in 2008 is recorded primarily in the 2009 gap estimate.

standard deviations for the United States, Japan, and Sweden; more than four standard deviations for the euro area; and more than *six* standard deviations for the United Kingdom. These numbers testify to the scale of the shock that has hit the world economy. The shock, and the associated policy response, are also reflected in the major revisions that occurred between 2007 and 2009 in forecasts in every economy of inflation and short-term interest rate (Figures 1 and 2).

2.2 Forward Guidance Measures

The reinforcement of current policy actions with signals about the future course of policy is at the heart of the forward guidance approach. As King (1994) observes, the rational expectations revolution highlighted the role of private sector expectations as a major conduit through which monetary policy affects aggregate demand.[5] King (1994) further notes that the U.S. monetary policy tightening sequence of 1994–95–at the onset of which the FOMC started announcing the target federal funds rate–triggered long-term interest rate responses that were "related in important ways to expectations about future policy."

The FOMC statement issued in August 2003 marked the first occasion on which the Federal Reserve gave forward guidance about the likely evolution of its funds rate target. At that point, the FOMC maintained the target federal funds rate at 1 percent and stated that "the risk of inflation being undesirably low is likely to be the predominant concern for the future. In these circumstances, the Committee believes that policy accommodation can be maintained for a considerable period." The minutes of that FOMC meeting indicated that "While the Committee could not commit itself to a particular policy course over time, many of the members referred to the likelihood that the Committee would want to keep policy accommodative for a longer period than had been the practice in past periods of accelerating economic activity." Although the description of the inflation outlook varied in subsequent FOMC statements, the "considerable period" language was retained through the end of 2003.

From May 2004 through the end of 2005, FOMC statements indicated that "the Committee believes that policy accommodation can be removed at a pace that is likely to be

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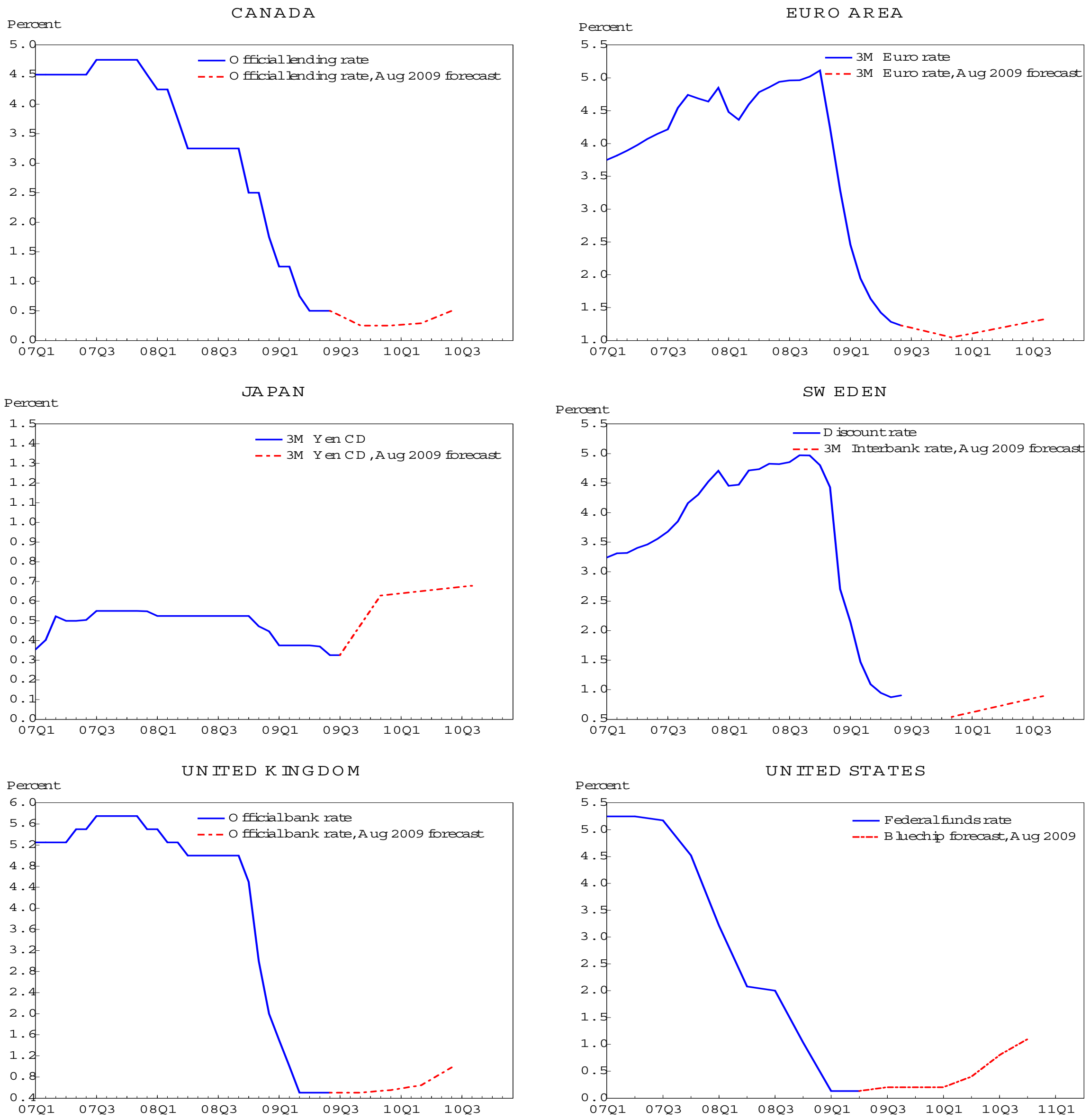
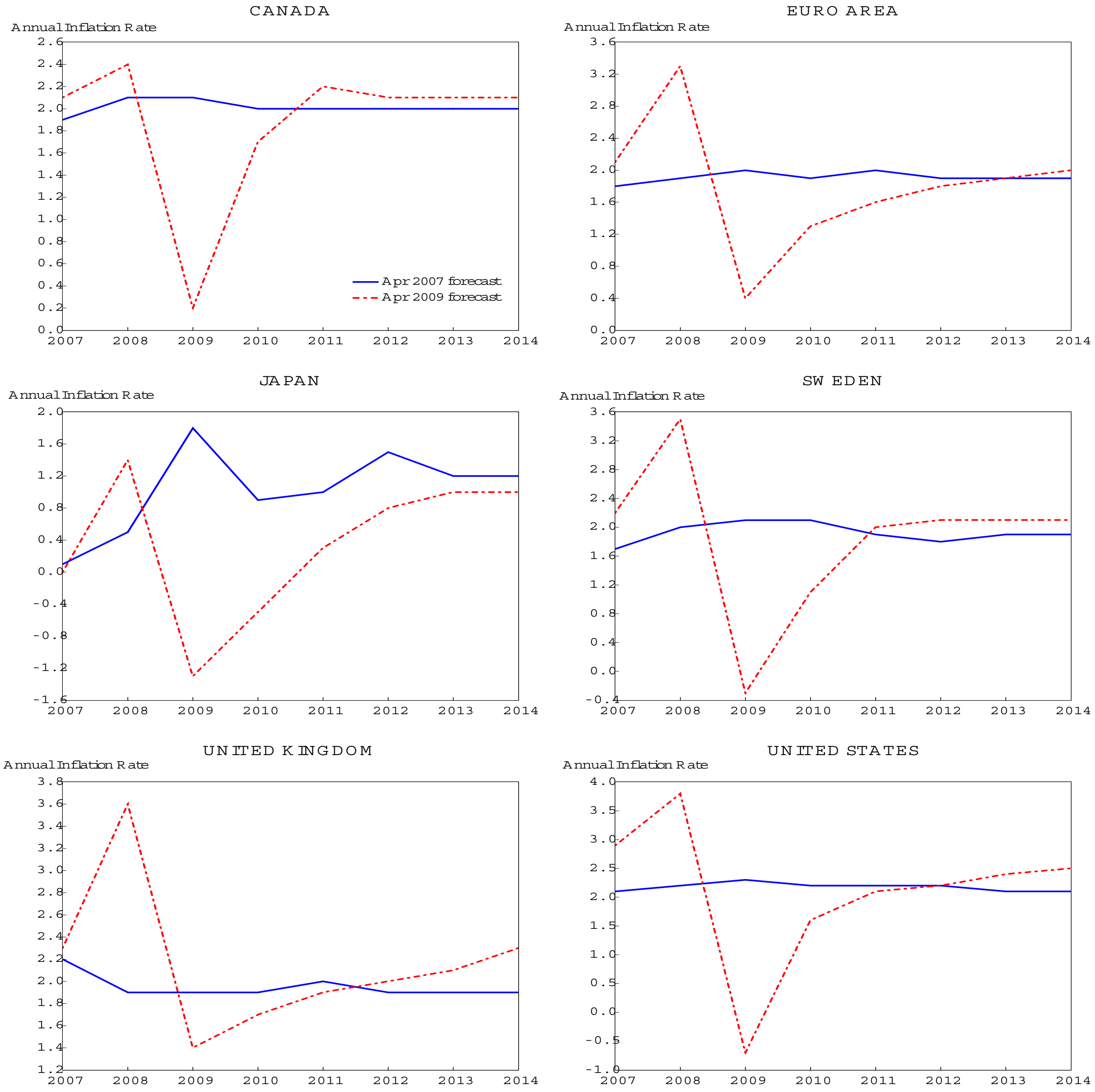
[5] The dual insights that long-term interest rate behavior reflects expectations about future policy, and that official signals about future short-term interest-rate policy can contribute to economic stabilization, are a longstanding feature of discussions of monetary policy, predating even the use of forward-looking models in macroeconomics; see e.g., Radcliffe Committee (1959, para. 447).

Figure 1: Interest Rate Expectations



Source: US forecast: Blue Chip Financial Forecasts, Vol. 28, No. 8, August 1, 2009; forecasts of federal funds rate. Other countries: Consensus forecasts, August 10, 2009 (Japan, three-month yen CD; euro area 3-month interest rates; U.K., bank rate; Canada, overnight lending rate; Sweden, 3-month interbank rate.)

Figure 2: Inflation Expectations



Source: Consensus Economics, April 2008 and April 2009; long-term forecasts of consumer prices in United States, Japan, euro area, U.K., Canada, and Sweden.

measured." The FOMC also underscored the conditional nature of this forward guidance by stating that "...the Committee will respond to changes in economic prospects as needed to fulfill its obligation to maintain price stability." This conditionality was introduced in June 2004–the point at which the FOMC began steadily raising the target federal funds rate by 25 basis points per meeting until this rate reached 5.25 percent in June 2006.[6]

Since December 2008, the FOMC has maintained a target range of 0 to 1/4 percent for the funds rate and has included forward guidance in each of its statements. The December 2008 FOMC statement referred to the likelihood that economic conditions would "warrant exceptionally low levels of the federal funds rate for some time," and in March 2009 this language was adjusted to refer to keeping rates low "for an extended period." Over the past year, the Sveriges Riksbank and the Bank of Canada have also included forward guidance in their policy communications.[7] For example, the July 2009 *Monetary Policy Report* of the Sveriges Riksbank states that "The repo rate will not be raised again until the second half of 2010." (p. 7)

As shown in Figure 1, surveys of professional forecasters indicate that short-term nominal interest rates are expected to follow a fairly shallow path in the United States, Canada, and Sweden. It should be noted, however, that the anticipated path of short-term rates is fairly similar for three other industrial economies–that is, the Euro Area, Japan, and the United Kingdom–where central bank communications have *not* emphasized forward policy guidance.

Figure 2 depicts professional forecasters' inflation expectations for each of these six industrial economies. Longer-run inflation expectations appear to have remained relatively well-anchored despite the global economic downturn; that is, consumer inflation in each economy is projected to settle in over the longer run at rates close to those which were anticipated in spring 2007, prior to the onset of the financial crisis. Nonetheless, as noted by Walsh (2009), the medium-term trajectory for inflation does *not* seem to be consistent with the existing literature on optimal policy under commitment, which prescribes an inflation path that rises *above* the long-run goal and remains elevated for an extended period.

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[6] From January-April 2004, the FOMC maintained an unchanged funds rate target of 1 percent and stated that "With inflation low and resource use slack, the Committee believes that it can be patient in removing its policy accommodation." The conditionality language was included in each FOMC statement from June 2004 through November 2005.

[7] For further discussion, see Bean (2009).

2.3 Nontraditional Monetary Policies

We use the term "nontraditional monetary policies" to refer to monetary policy operations in additional assets beyond the "traditional" focus on short-term government securities. As noted in the introduction, a variety of nontraditional policies have been deployed since the onset of the economic downturn.

In late November 2008, the Federal Reserve announced that it would use "all available tools" to promote economic recovery and preserve price stability. At that time and in subsequent FOMC announcements, the Federal Reserve indicated that it would "provide support to mortgage lending and housing markets" by purchasing up to \$1.25 trillion in agency mortgage-backed securities and up to \$200 billion in agency debt. In March 2009, the FOMC announced purchases of up to \$300 billion in Treasury securities "to help improve conditions in private credit markets." The Federal Reserve also launched the Term Asset-Backed Securities Loan Facility [TALF] "to facilitate the extension of credit to households and small businesses."

A number of other central banks have also initiated and expanded their nontraditional policy measures over the past year. In December 2008, the Bank of Japan announced that it would accelerate the pace of its purchases of long-term Japanese government bonds (JGBs) and that these purchases would be expanded to include 30-year bonds and inflation-indexed government securities. In early March 2009, the Bank of England announced that it would engage in unsterilized purchases of up to 75 thousand million in U.K. government securities; about two months later, the upper limit on those purchases was increased to 125 thousand million. Finally, the European Central Bank also announced that Announcement of up to 60 billion purchase of euro-denominated covered bonds; purchases scheduled for July 2009-June 2010

The remainder of this paper will consider the limits to forward guidance and the extent to which such limits may serve as an significant rationale for nontraditional policy measures. Although our paper is not aimed at modelling or quantifying the stimulative effects of nontraditional policies, a few brief comments may be helpful at this point.

First, while economists have become accustomed to working with models that feature traditional short-term interest rate policy, nontraditional policies are not new from a longer-term perspective; the notion that central bank operations in longer-term debt markets can affect long-term interest rates for a *given* path of expected short-term interest rates has a venerable history, both in central banking (e.g., Riefler, 1958) and the research literature (e.g., Modigliani and Sutch, 1966). The fact that nontraditional policies have no role in the modern consensus model may reflect a gap in the modern research agenda rather than an inherent problem with these policies.

Second, while the empirical evidence in support of nontraditional policies has been questioned—for example, Goodhart (1992, p. 327) states that "studies of the effect of relative debt supplies (at the long, medium and short end) on the yield curve have not found any strong, significant effect," and similar observations have been made by Woodford (1999) and Walsh (2004, 2009)—the 2000 U.S. Treasury refinancing produced favorable evidence; see Bernanke, Reinhart, and Sack (2004). Moreover, as Sims (1992, p.975) observes, while many economists view monetary policy's effects solely through the nominal short-term interest rate channel, "the profession as a whole has no clear answer to the question of the size and nature of the effects of monetary policy on aggregate activity." Judgments on the empirical support for nontraditional policies should therefore remain open.

Third, the argument about nontraditional policies' effects is a *ceteris paribus* argument. Central bank purchases of long-term debt might provide downward *pressure* on long-term rates, and thereby contribute to an improvement in the economic outlook; but an improved outlook will in turn tend to raise long-term rates, and is thought to have done so since early 2009.[8] For the United Kingdom, Bean (2009) argues that the *ceteris paribus* effect of the Bank of England's long-term purchases has been substantial, contending that long-term "yields appear to be some 50-75 basis points lower than they would otherwise be." If estimates of effects if this order of magnitude endure, then nontraditional monetary policies would appear capable of making a major contribution to economic stabilization by influencing private credit conditions.

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[8] An early articulation of this point appeared in Friedman and Meiselman (1963, p. 221). They argued that a zero net effect of a monetary injection on market interest rates could be a reflection of the power of monetary policy, since anticipation of higher spending streams from the monetary stimulus could generate upward, offsetting pressure on interest rates.

3 Methodology

In this section, we provide a brief overview of the model economy and then characterize the optimal policy under commitment.

3.1 The Model Economy

Our analysis focuses on the same stylized New Keynesian model that has been used in many previous studies. The full nonlinear model is presented in the Appendix. With a non-distorted steady state, the loglinearized model has only two behavioral equations: a forward-looking IS equation and a forward-looking New Keynesian Phillips curve.[9] As in Woodford (1999), it is convenient to express the IS equation in terms of output gaps and real interest rate gaps:

$$x_t = E_t\{x_{t+1}\} - \sigma E_t\{i_t - \pi_{t+1} - r_t^n\}, \tag{1}$$

where x_t represents the output gap, i_t is the net nominal interest rate, π_t is inflation, r_t^n represents the natural real rate of interest and, and $\sigma > 0$ is the elasticity of intertemporal substitution.[10]

With Calvo-style staggered price setting, the New Keynesian Phillips curve has the following form:

$$\pi_t = \beta \; E_t\{\pi_{t+1}\} + \kappa \; x_t \tag{2}$$

where $0 < \beta < 1$ and $\kappa > 0$.

Our parameterization of the model makes reference to the elasticities σ and κ in these relations, as well as the process followed by the natural rate, r_t^n.

3.2 The Ramsey Policy

When it comes to deriving the optimality conditions for the policymaker, our initial analysis is modeled on the approach of Khan, King, and Wolman (2003) and therefore pertains to the nonlinear economy.[11] A Ramsey policymaker maximizes conditional intertemporal welfare

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[9] See Kerr and King (1996) and King (2000) for further discussion.

[10] As we show in the appendix, movements in the natural rate can be related to underlying exogenous variations in technology and government spending.

[11] Technical details are outlined in the appendix.

of households from the viewpoint of period zero, subject to specified implementability conditions drawn from the structure of the model. We specifically augment the implementability constraints with a condition embodying the possibility of a zero lower bound on the short-term nominal interest rate. This becomes a binding constraint in our quantitative analysis when there is a sudden decline in the natural real rate of interest to far below its steady-state value.

We assume that private employment is subsidized (by the subsidy τ) in a way that extinguishes steady-state effects on the aggregate markup that would otherwise arise from firms' monopolistically competitive character. The optimal resource allocation is therefore attainable at the nonstochastic, zero-inflation steady state. While the markup is subsidized away on average, temporary fluctuations in the markup arise from gradual price adjustment. Their presence implies that firms' rules for setting goods prices become binding constraints on the Ramsey policymaker's optimization.

In the deterministic case considered here, the behavior of the model economy under the Ramsey policy can be demarcated into two distinct phases: (1) the initial sequence of periods where the policy instrument–that is, the short-term nominal interest rate– is at the ZLB; and (2) the subsequent sequence of periods where the policy rate becomes positive and eventually returns to its steady-state value. During the first phase, the Lagrange multiplier associated with the household Euler equation for consumption (i.e, the IS equation) exceeds zero, reflecting the fact that the ZLB constrains the optimal path of the policy instrument. In contrast, this Lagrange multiplier is continuously equal to zero during the second phase, when the ZLB is no longer a binding constraint.

Thus, for the deterministic case, it is natural to consider a piecewise-linear approximation to the behavior of the model economy. In particular, having obtained the nonlinear optimality conditions, we can compute a linear approximation to the economy for the first set of periods in which the policy rate is equal to zero and the Lagrange multiplier on the IS equation is positive, and a second linear approximation for the set of periods where the policy rate exceeds zero and the Lagrange multiplier is equal to zero. Moreover, an iterative guess-and-verify method can be used to determine the "lift-off date", that is, the first period in which the policy rate rises above the ZLB.

3.3 An Equivalence Result

Benigno and Woodford (2008) showed that it is possible, under regularity conditions that are straightforward to check, to derive a problem with linear constraints and a quadratic objective that approximates the exact problem for a general class of nonlinear optimal policy problems involving forward-looking constraints (such as the Euler equations that are typically present as structural equations in DSGE models). Specifically, the solution to the LQ approximate problem represents a local linear approximation to the optimal policy for the exact model in the case that stochastic disturbances are small enough, given the second-order conditions that must be satisfied in order for the LQ problem to have a solution.

In this paper, we augment their result to models with zero lower bound on the nominal interest rate. But we note that our analysis focuses on the perfect foresight equilibrium dynamics. The reason why we do this is that the LQ approximation requires piecewise linear approximations of equilibrium conditions.

Assumption 1 There exists a unique solution to the exact optimal policy problem that allows for the possibility of the zero lower bound on the nominal interest rate, which is described above.

Assumption 2 The paths of exogenous disturbances to the model are perfectly known at period 0. The sizes of exogenous disturbances are small enough not to harm validity of local approximations of the model.

Assumption 3 The second-order conditions that must be satisfied in order for the LQ problem to have a solution provided in Benigno and Woodford (2008).

Proposition 1 Suppose that assumptions 1, 2, and 3 hold. Then, there is a problem with linear constraints and a quadratic objective that approximates the exact problem. The solution to the LQ approximate problem represents a local linear approximation to the optimal policy for the exact model.

4 Quantifying the Limitations of Forward Guidance

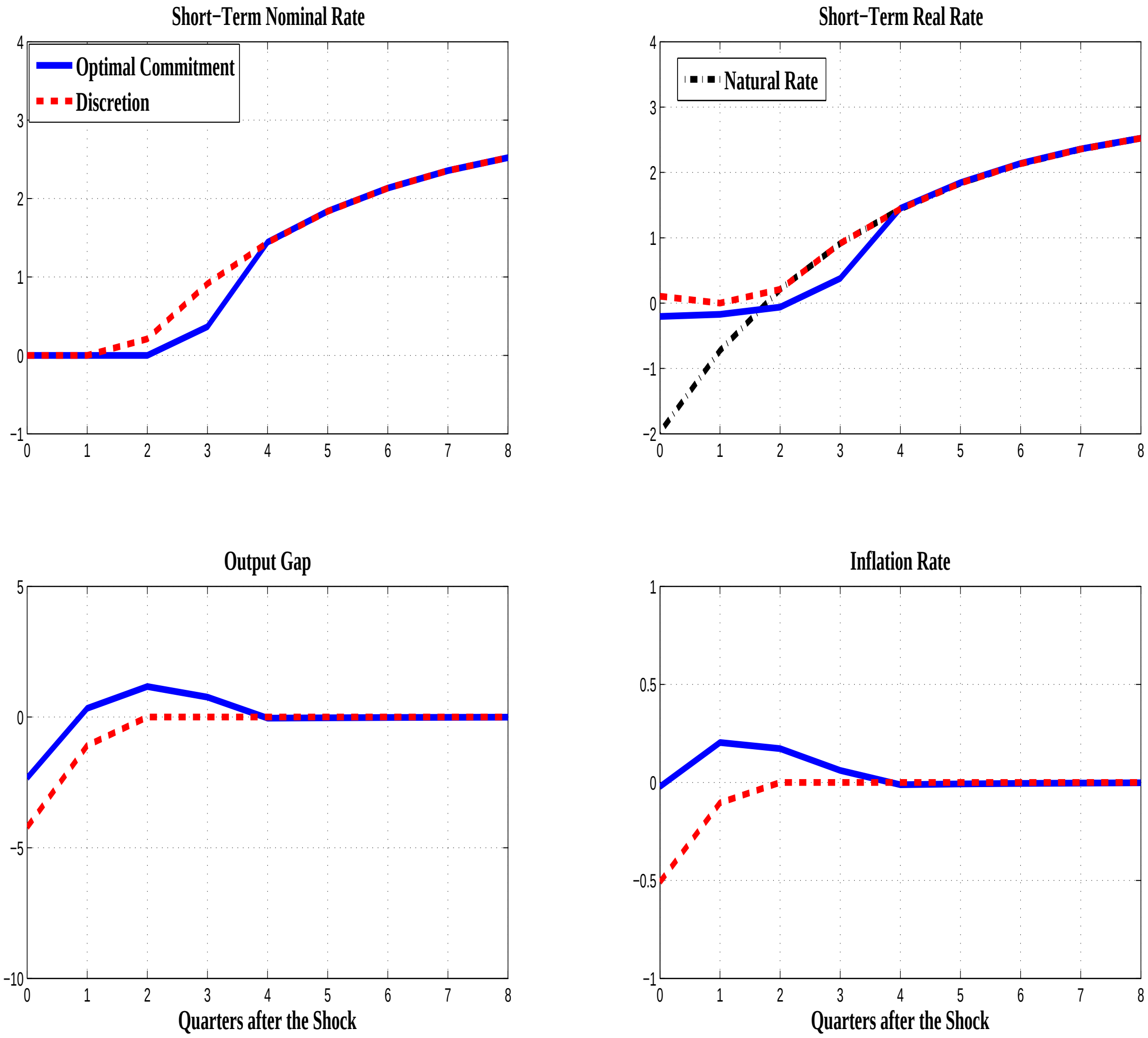
To quantify the benefits and limitations of forward guidance at the ZLB, we examine scenarios in which an exogenous decline in aggregate demand pushes the natural real rate of interest below zero. which, in turn, prompts policymakers to cut the short-term nominal interest rate to zero. In this section, we follow Woodford (2003) in calibrating the structural parameters ($\beta = 0.9925$, $\kappa = 0.024$, and $\sigma = 6$), and we consider two specifications for the natural rate shock: an autoregressive shock of modest size and persistence–as in Adam and Billi (2003, 2006) and Nakov (2008)–that may be viewed as characteristic of the "Great Moderation" era; and a more severe and persistent shock that can be interpreted as representing a "Great Recession"-style episode. The subsequent section will consider alternative values of the interest elasticity parameter (σ) and will examine the case in which the shock follows a two-state Markov process, as in Eggertsson and Woodford (2003).

4.1 Macroeconomic Outcomes under the Optimal Policy

In Figure 3 we consider a "Great Moderation"-style shock of about 5 percent to the natural rate that fades out within a few quarters; that is, the shock follows an AR(1) process with parameter $\rho = 0.75$. As shown in the upper-left panel, the optimal policy under discretion is only constrained by the ZLB for two quarters, whereas the optimal policy under commitment keeps the short-term nominal interest rate at zero for an additional quarter. By keeping the nominal interest rate at zero for somewhat longer period than under discretion, the optimal commitment policy dampens the impact on the output gap. The deviations of inflation from zero are consistently mild, and these deviations cumulate to about zero over a couple of years, so that the inflation path closely resembles that implied by a simple rule with a constant price level target.

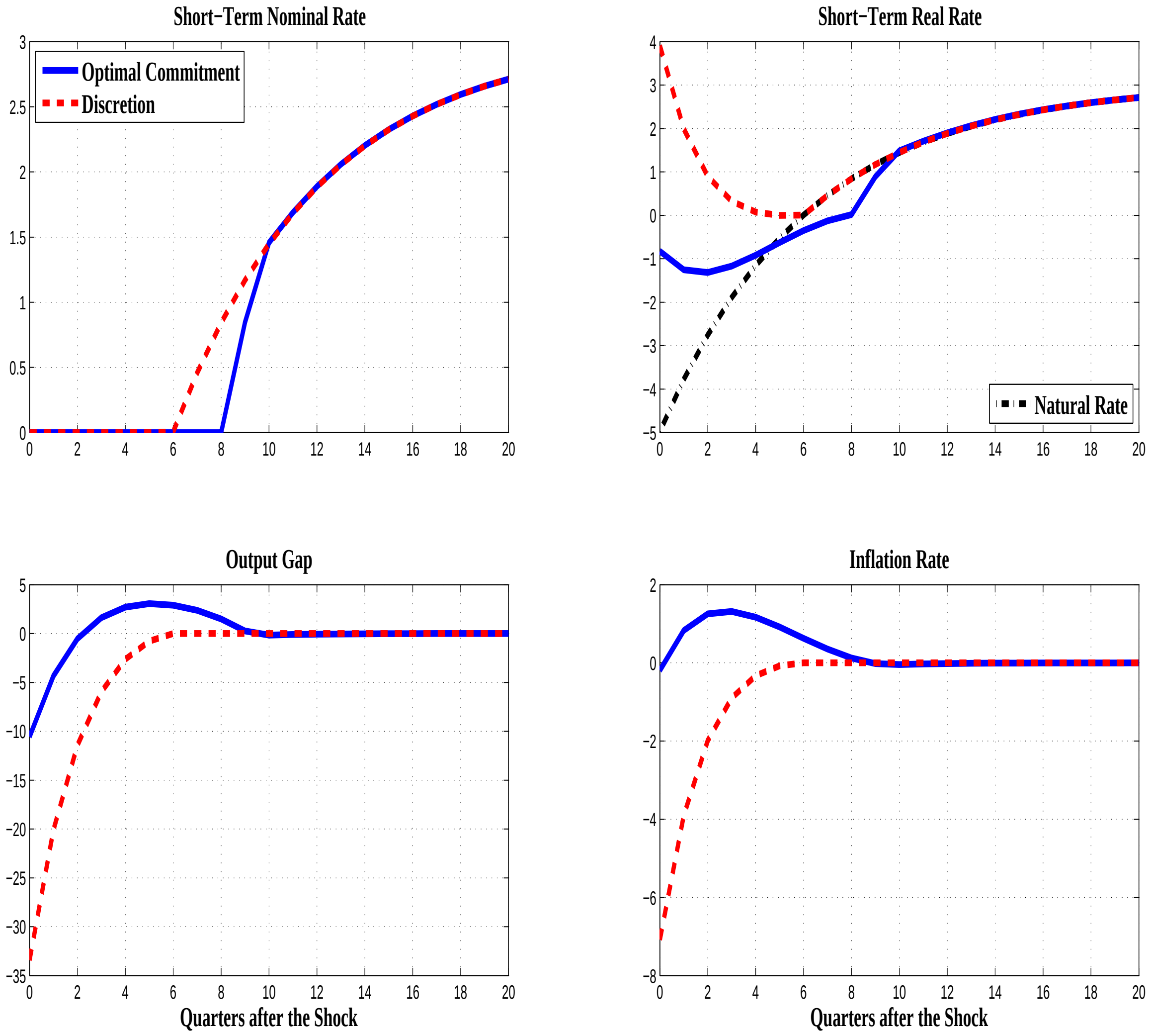
It is, however, the case of a large AR(1) shock, shown in Figure 4, that brings home the limitations of forward guidance. Here, even promises of fluctuations in inflation for several additional quarters fails to prevent the natural rate shock from causing a deep output gap at time zero. Moreover, a substantial rise in inflation is required to push down real interest rates and thereby avoid an even steeper decline in output such as that observed under the discretionary policy.

Figure 3: “Great Moderation”-Style Shock



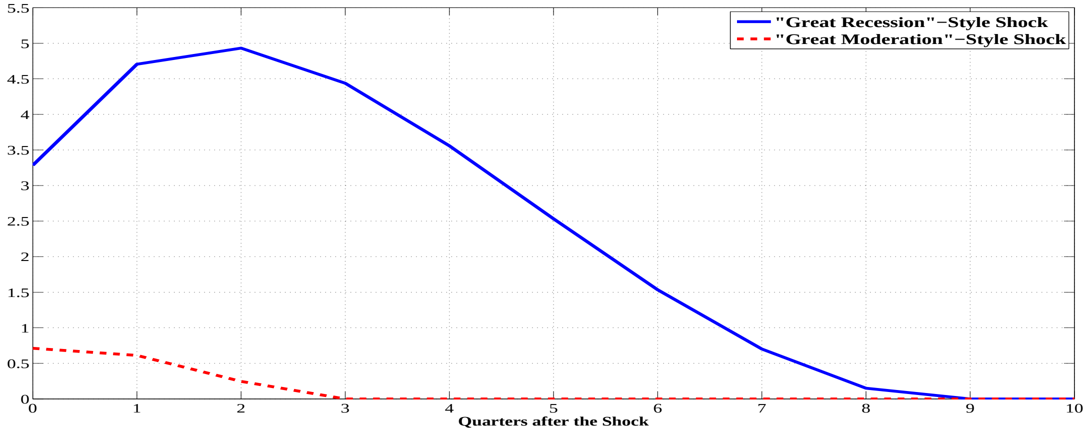
Note: This simulation was performed using the baseline calibration of $\beta = 0.9925$, $\kappa = 0.024$, and $\sigma = 6$. The natural rate shock follows an AR(1) process with first-order autocorrelation $\rho = 0.75$. The short-term nominal interest rate, the short-term real interest rate, and the inflation rate are each expressed at annual rates in percent; the output gap is expressed in percentage points.

Figure 4: “Great Recession”-Style Shock



Note: This simulation was performed using the baseline calibration of $\beta = 0.9925$, $\kappa = 0.024$, and $\sigma = 6$. The natural rate shock follows an AR(1) process with first-order autocorrelation $\rho = 0.85$. The short-term nominal interest rate, the short-term real interest rate, and the inflation rate are each expressed at annual rates in percent; the output gap is expressed in percentage points.

Figure 5: The Lagrange Multiplier on the Dynamic IS Equation



Note: This simulation was performed using the baseline calibration of $\beta = 0.9925$, $\kappa = 0.024$, and $\sigma = 6$. The natural rate follows an AR(1) process with first-order autocorrelation $\rho = 0.75$ for the "Great Moderation"-style shock and $\rho = 0.85$ for the "Great Recession"-style shock. The Lagrange multiplier is depicted using a scaling factor of 10^{-5}.

For each of these two shocks, Figure 5 depicts the corresponding trajectories for the Lagrange multiplier on the dynamic IS equation. As discussed in the previous section, this Lagrange multiplier is positive during periods when the ZLB is an active constraint and falls to zero once the ZLB no longer constrains the optimal setting of the short-term nominal interest rate. Thus, the magnitude of this Lagrange multiplier provides a useful measure of the extent to which the ZLB reduces social welfare at each point in time. For the "Great Moderation"-style shock, these welfare costs are very small and transitory, reflecting the effectiveness of forward guidance in providing stabilization outcomes that are nearly as good as in the absence of the ZLB. In contrast, the Lagrange multiplier is roughly an order of magnitude larger–and much more persistent–for the "Great Recession"-style shock, thereby providing further perspective on the limitations of forward guidance in this case.

4.2 The Performance of a Constant Price-Level Targeting Rule

Now we consider the extent to which the optimal commitment policy can be replicated by a constant price-level targeting rule, such as the one proposed by Eggertsson and Woodford (2003):

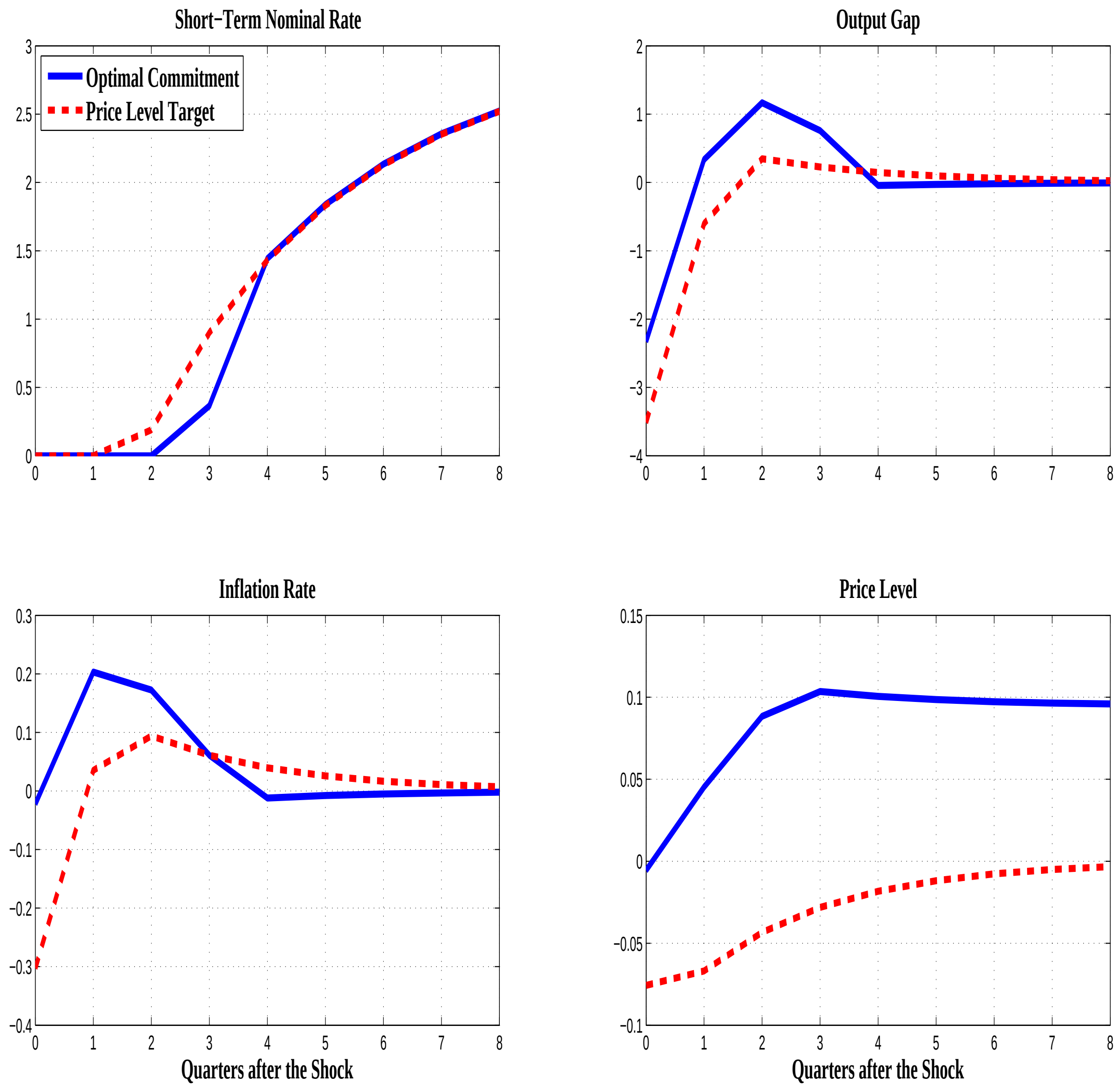
$$p_t - \frac{\lambda}{\kappa} x_t = p* \tag{3}$$

where p_t denotes the logarithm of the price level and $p*$ denotes the price-level target (which is a time-invariant constant). As noted by Eggertsson and Woodford (2003), this rule does not no involve any special provisos related to the ZLB and hence might be simpler to communicate than the optimal commitment policy.

Figure 6 depicts the performance of the constant price-level targeting rule for the "Great Moderation"-style shock. In this case, the simple rule generates macroeconomic outcomes that are nearly as good as those obtained under the optimal commitment policy, that is, the output gap and inflation rate exhibit only slightly higher variability.

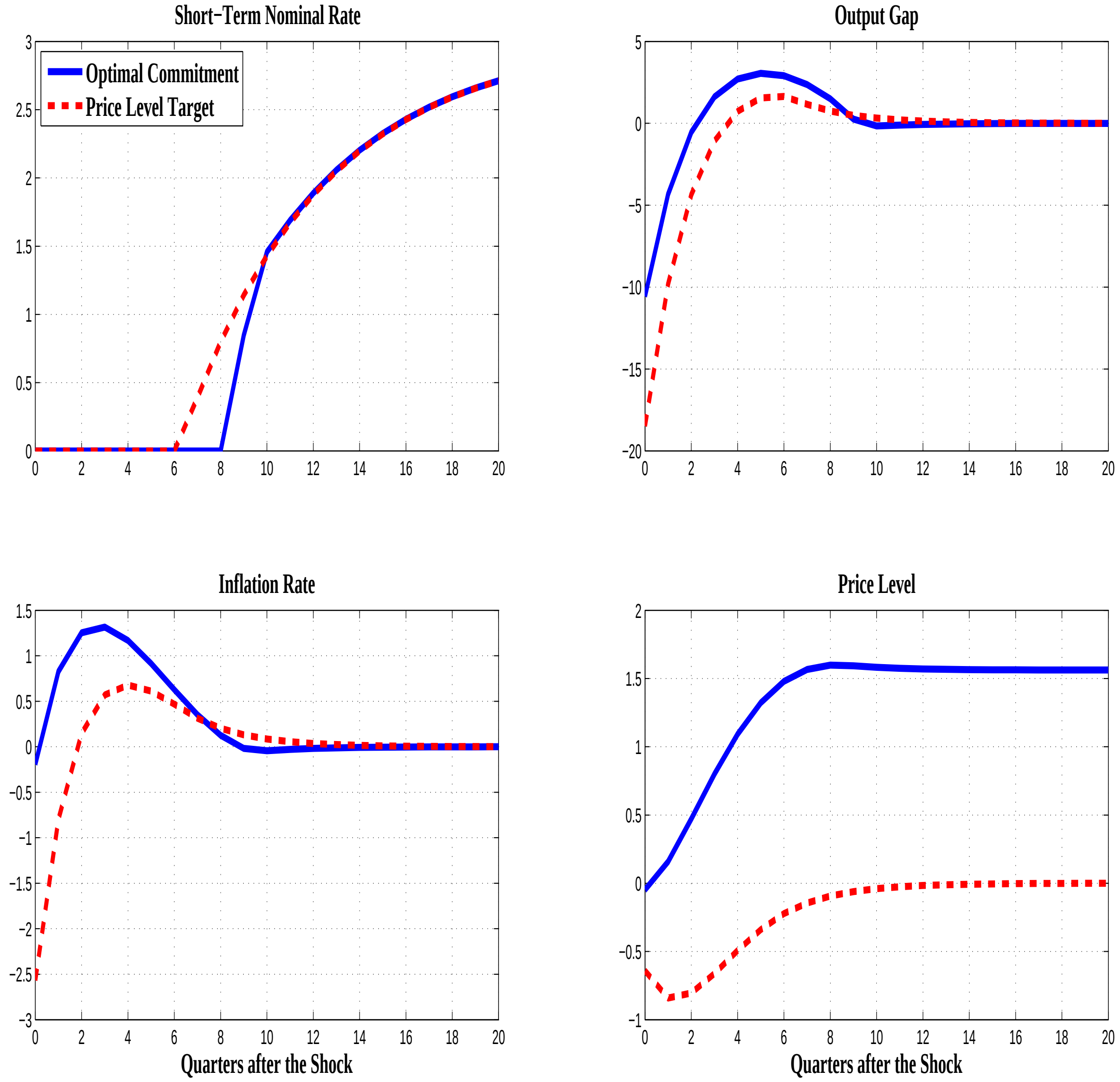
As shown in Figure 7, however, the stabilization performance of the constant price-level targeting rule in response to the "Great Recession"-style shock is dramatically inferior to that of the optimal commitment policy. In this case, the optimal policy prescribes a persistently elevated inflation rate–roughly similar to the policy strategy advocated by Krugman (1998)–that pushes down the *ex ante* real interest rate and thereby dampens the initial impact of the shock. In contrast, the constant price level targeting rule generates an initial phase of deflation–with a much steeper drop in output than under the optimal policy–and a subsequent phase of positive inflation that eventually brings the price level back to target.

Figure 6: “Great Moderation”-Style Shock with a Constant Price-Level Targeting Rule



Note: This simulation was performed using the baseline calibration of $\beta = 0.9925$, $\kappa = 0.024$, and $\sigma = 6$. The natural rate shock follows an AR(1) process with first-order autocorrelation $\rho = 0.75$. The short-term nominal interest rate and the inflation rate are each expressed at annual rates in percent; the output gap is expressed in percentage points; and the price level is expressed as the percent deviation from the initial price level prior to the onset of the shock. in percentage

Figure 7: “Great Recession”-Style Shock with a Constant Price-Level Targeting Rule


Short-Term Nominal Rate

Output Gap

Inflation Rate

Price Level

Note: This simulation was performed using the baseline calibration of $\beta = 0.9925$, $\kappa = 0.024$, and $\sigma = 6$. The natural rate shock follows an AR(1) process with first-order autocorrelation $\rho = 0.85$. The short-term nominal interest rate and the inflation rate are each expressed at annual rates in percent; the output gap is expressed in percentage points; and the price level is expressed as the percent deviation from the initial price level prior to the onset of the shock.

5 Sensitivity Analysis

In this section, we perform further sensitivity analysis with respect to the interest elasticity of aggregate demand, and we examine the case in which the natural rate shock follows a two-state Markov process, as in Eggertsson and Woodford (2003), rather than an AR(1) process.

5.1 The Interest Elasticity of Aggregate Demand

In the foregoing analysis, we have used the benchmark calibration of Woodford (2003), in which the interest elasticity parameter $\sigma = 6$. Rotemberg and Woodford (1997) obtained this parameter value by applying minimum-distance estimation to a small stylized New Keynesian model, using U.S. aggregate time series data (that is, real GDP, the GDP price inflation rate, and the federal funds rate).[12] Of course, numerous other studies have estimated the slope of the dynamic IS equation using a variety of empirical procedures. For example, Amato and Laubach (2003) obtained the estimate $\sigma = 4$ using the same estimation procedure as Rotemberg and Woodford (1997) but allowing for nominal rigidity in wages as well as prices. In contrast, Eggertsson and Woodford (2003) specified a much smaller value of $\sigma = 0.5$, on the premise that a lower interest elasticity would tend to provide less scope for forward guidance.

Figure 8 depicts the optimal policy path and associated macroeconomic outcomes for different values of the parameter σ for the "Great Recession"-style shock that was considered in the previous section. Evidently, the shortcomings of forward guidance are roughly similar for either empirically plausible specification of the interest elasticity ($\sigma = 4$ or $\sigma = 6$). In contrast, when the interest elasticity is very low ($\sigma = 0.5$), the optimal commitment policy delivers virtually impeccable stabilization outcomes that are roughly comparable to the results obtained by Eggertsson and Woodford (2003).

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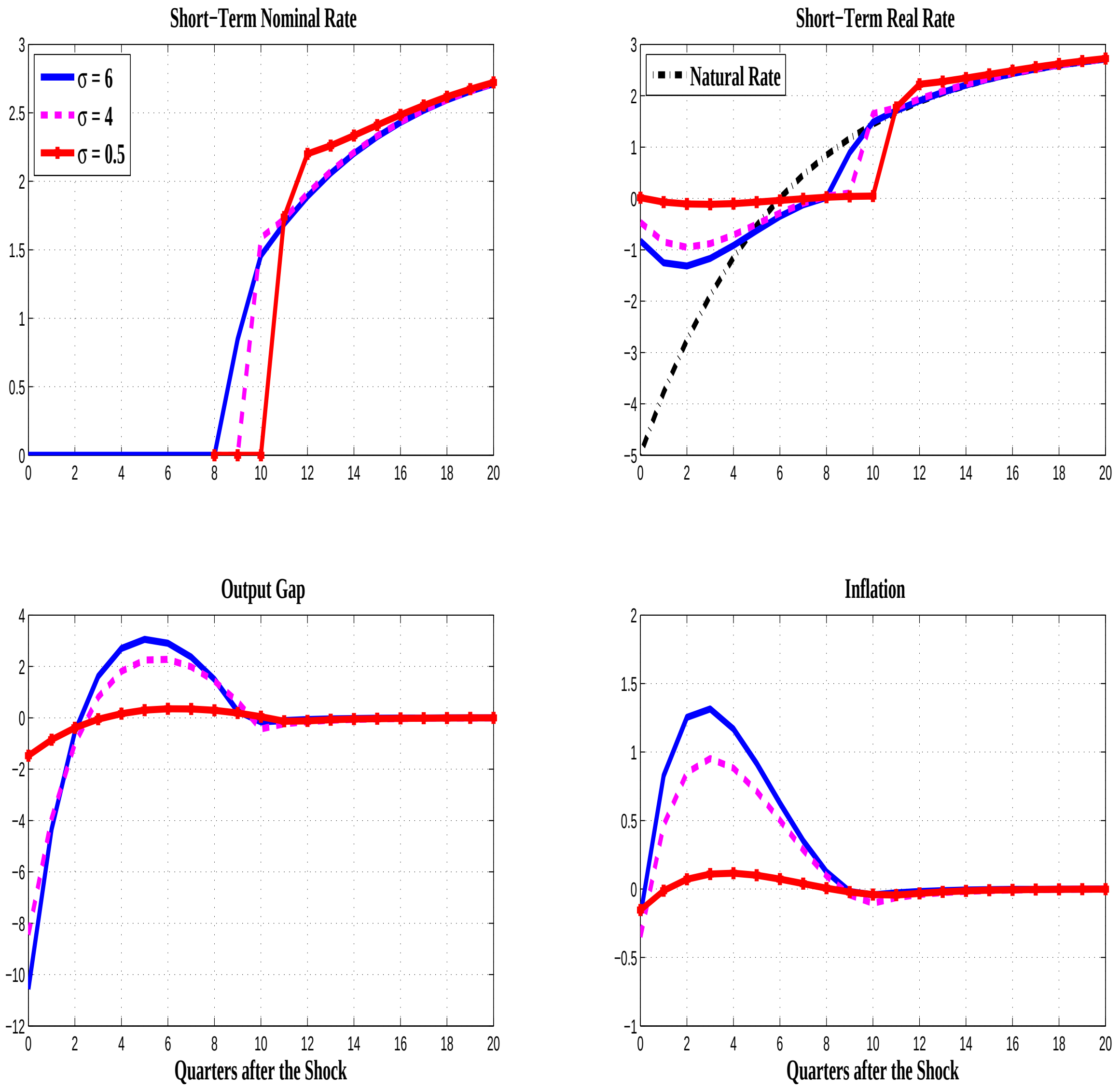
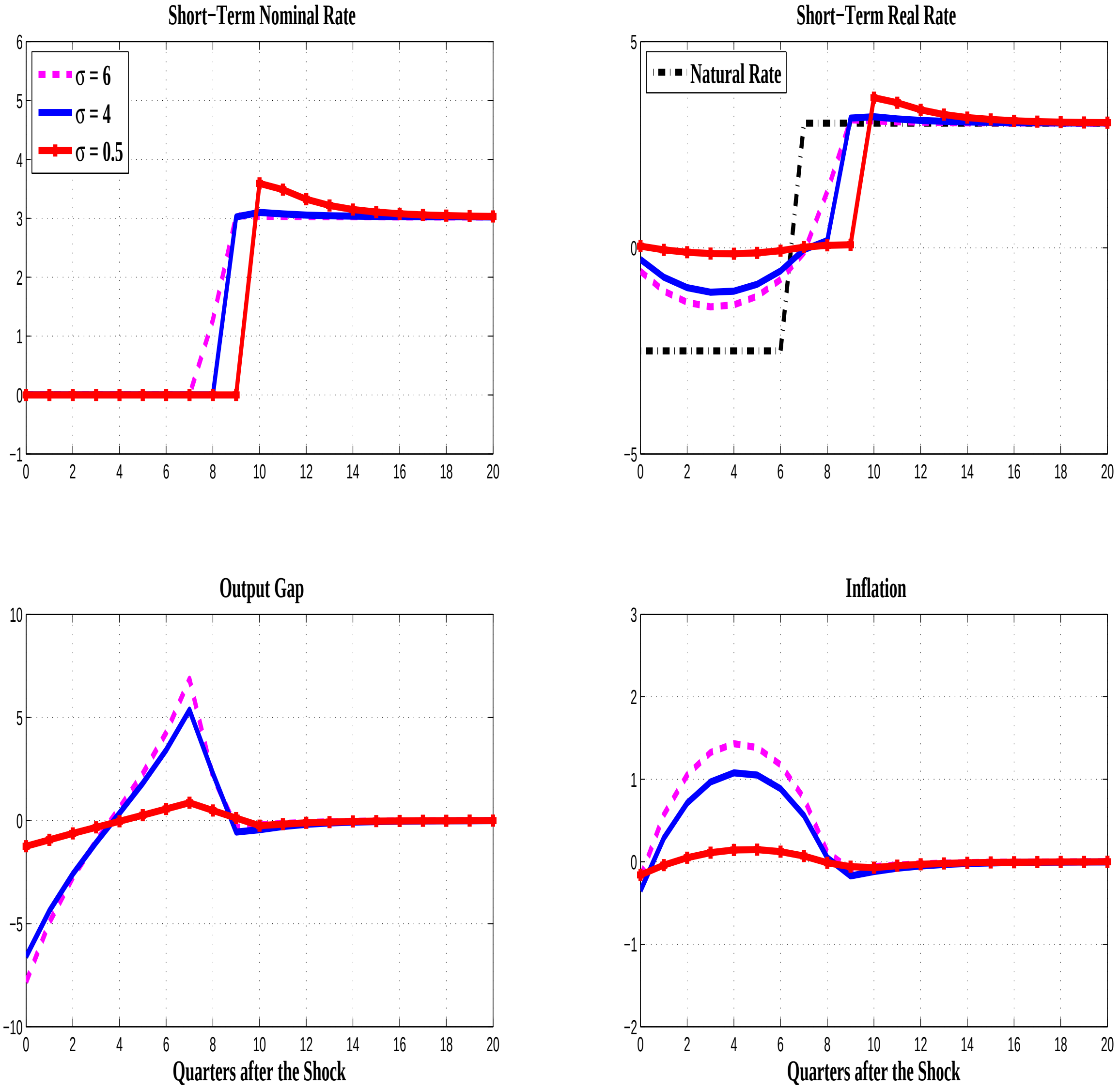
[12] It should be noted that Rotemberg and Woodford (1997) and Woodford (2003) used the symbol σ to denote the degree of relative risk aversion, whereas we follow the notation of Eggertsson and Woodford (2003) in using σ to denote the interest elasticity, that is, the inverse of the risk aversion parameter. Thus, the value of $\sigma = 6$ used in this paper corresponds to the risk aversion parameter estimate of 0.157 obtained by Rotemberg and Woodford (1997) and used in the benchmark calibration of Woodford (2003).

Figure 8: The Interest Elasticity of Aggregate Demand



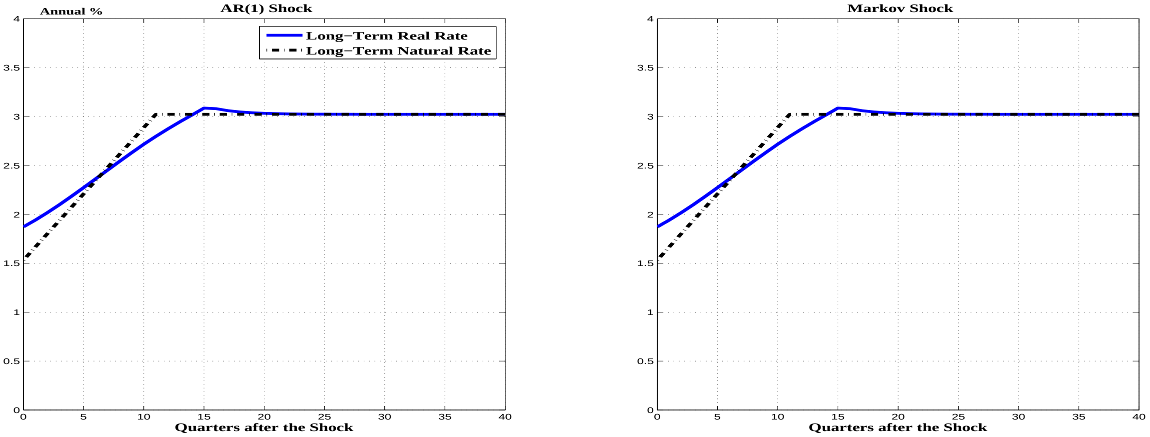
Note: This simulation was performed using the baseline calibration of $\beta = 0.9925$ and $\kappa = 0.024$, with three alternative values of σ (0.5, 1, and 6). The natural rate shock follows an AR(1) process with a persistence parameter of 0.875

Figure 9: The Interest Elasticity and Markov Shocks



Note: This simulation was performed using the baseline calibration of $\beta = 0.9925$ and $\kappa = 0.024$, with three alternative values of σ (0.5, 1, and 6). The natural rate shock follows a two-state Markov process.

Figure 10: Long-Term Real Rates: AR(1) vs. Markov Shocks



Note: This simulation was performed using the baseline calibration of $\beta = 0.9925$, $\kappa = 0.024$, and $\sigma = 1$.

5.2 Two-State Markov Shocks

The analysis of Eggertsson and Woodford (2003) focused on natural rate shocks generated by a two-state Markov process. To gauge the implications of this specification, let us now consider a Markov shock with magnitude and persistence comparable to the "Great Recession"-style AR(1) shock that we have been considering heretofore. In particular, we shall assume that the Markov shock reduces the natural real rate by about 4 percent and lasts for seven quarters.

As shown in Figure 9, the implications of σ are roughly similar for the case of a Markov shock as for the case of an AR(1) shock. In particular, the optimal commitment policy generates virtually impeccable stabilization outcomes when $\sigma = 0.5$–the calibration used by Eggertson and Woodford (2003)–but are much less appealing for empirically reasonable values of this parameter ($\sigma = 4$ or $\sigma = 6$). Although not shown here, the considerations related to a constant price-level targeting rule are also essentially the same for a two-state Markov shock as for an AR(1) shock. The crucial issue for stabilization performance is *not* the shape of the shock trajectory, but the magnitude and persistence of the natural rate shock (that is, drawn from the "Great Moderation" era vs. representing a "Great Recession"-style episode) and the interest sensitivity of aggregate demand.

6 Uncertainty about the Pace of Recovery

Up to this point, we have considered cases where all agents are perfectly aware of the natural rate path from period zero onward. Eggertsson and Woodford (2003), however, also consider experiments where the two-state Markov process for the natural rate of interest is stochastic, in the sense that there is uncertainty regarding the period at which the natural real interest rate reverts to its steady-state value. As they show, this form of uncertainty actually magnifies the gains from forward guidance.

A variation on this approach, pursued here, is to formulate the shock as AR(1) and have uncertainty refer to the rate at which normal conditions resume; that is, to uncertainty about the AR(1) coefficient that propagates the shock. We conjecture that this form of uncertainty—in contrast to the uncertainty about Markov-switching in Eggertsson and Woodford's experiments—will tend to reduce the gains from forward guidance.

In order to examine this conjecture, we assume that the natural rate follows an AR(1) process with the possibility of a shift in the autoregressive parameter. Specifically, a negative shock to the natural real rate of interest strikes in period 0 and decays in each period at a constant rate ρ_1 through period 4:

$$r_0^n = r^n + \rho_1^t \epsilon_0$$

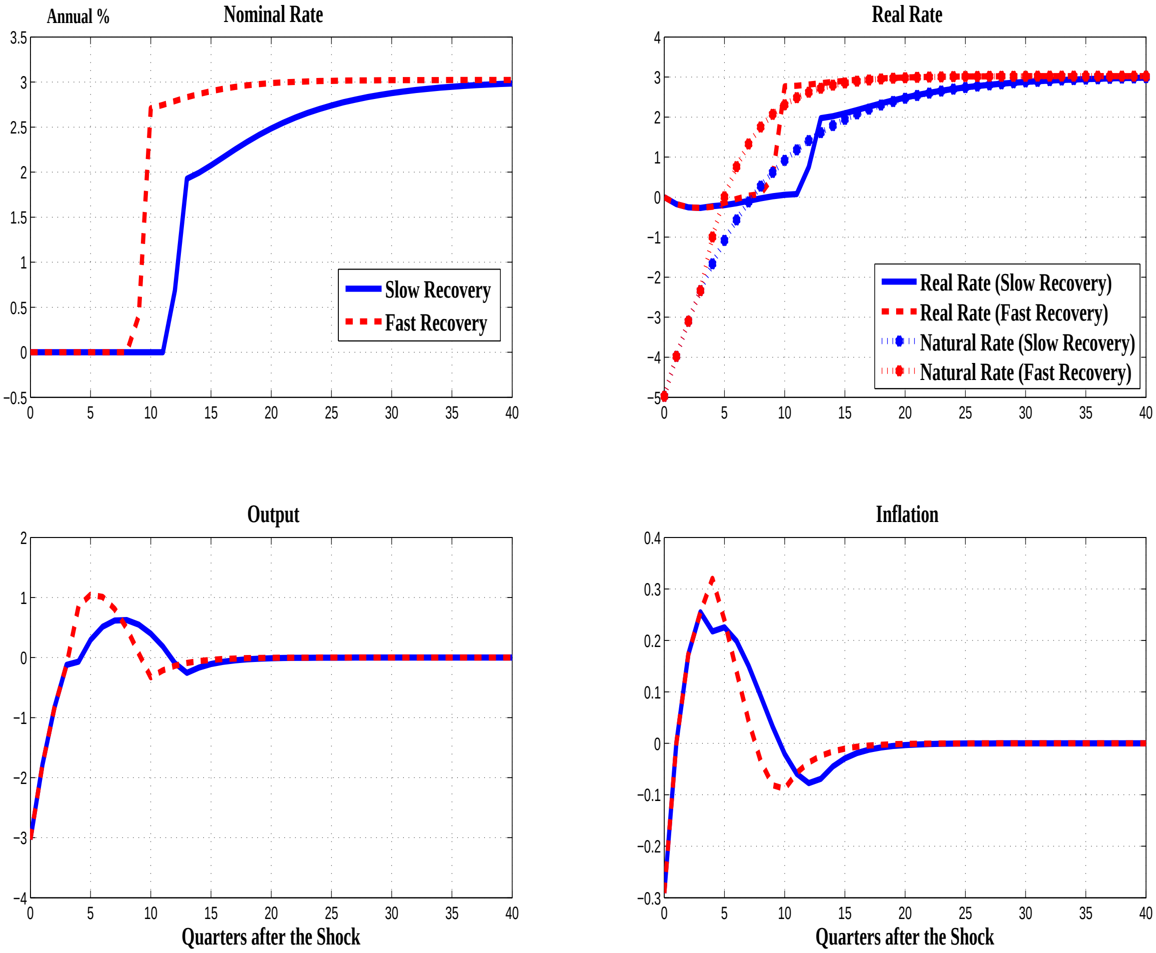
From period 5 onward, the AR parameter of the natural rate shock process is subject to an exogenous permanent shift, and takes a value of either $\rho_1 = 0.875$ or $\rho_2 = 0.75$ with a probability of 1/2 for each case.

$$\rho_t = \begin{cases} \rho_1 = 0.875 & \text{with } p = 0.5 \\ \rho_2 = 0.75 & \text{with } p = 0.5 \end{cases}$$

Hence, the timing of shifts in the persistence parameter is perfectly known at period 0 but the size of the change in the persistence parameter is not known until period 5. Conditional on this process, we use log-linearized optimal policy conditions in order to compute the lift-off period of the zero bound.

The motivation behind this type of stochastic shift in the decay rate of the natural rate shock is to analyze the impact of changes in the expected pace of recovery on the optimal commitment for forward guidance. Specifically, the optimal policy conditions imply that the

Figure 11: Stochastic Shifts in the Pace of Recovery



Note: This simulation was performed using the baseline calibration of $\beta = 0.9925$, $\kappa = 0.024$, $\sigma = 1$. The persistence of the aggregate demand shock is either $\rho_1 = 0.875$ or $\rho_2 = 0.75$).

central bank should rely less on the stimulus from forward guidance when the economy is expected to return to normal situations rapidly. Hence, the central bank should announce a shorter zero bound period when the economy will have a quick recovery at some point in the future. In particular, the optimal policy prescriptions from the Ramsey policy maker's problem is to make state-contingent commitments on the lift-off periods of the zero lower bound. For example, the central bank can announce different number of time periods of the zero lower bound conditional on each distinct realized path of future natural rates.

Figure 11 shows that the optimal commitment policy requires the central bank to make state-contingent announcements regarding the timing of the liftoff from the zero lower bound. For example, a downward shift in the persistence parameter is associated with an earlier departure from the zero bound; that is, the policy rate remains at the zero bound for nine quarters when the persistence parameter declines from $\rho_1 = 0.875$ to $\rho_2 = 0.75$ at period 4, whereas policy stays at the zero bound for twelve quarters when the persistence parameter does not shift from its initial value.

7 Conclusions

Our analysis in this paper has reconsidered optimal monetary policy in a New Keynesian model under conditions where a natural rate shock makes the zero lower bound on nominal interest rates a binding constraint on monetary policy. The existing literature has generally considered shocks of a size that are plausible in considering most countries' experience from 1984 to 2007, but are too low for thinking about the shock that preceded the 2008 downturn. However, when we consider a relatively large and persistent shock to the natural rate, we find that forward guidance alone is not sufficient to keep output close to potential and inflation close to the long-run goal. These results suggest that there could indeed be a role for nontraditional monetary policies as a complement to forward guidance. Thus, incorporating such mechanisms into a New Keynesian modelling framework is an important direction for further research.

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Appendix: The Optimal Policy Problem

The model economy corresponds to the baseline New Keynesian model and includes the staggered price-setting model of Calvo (1983), with consent given in any period only to a fraction of firms $(1-\alpha)$ to alter goods prices. These intermediate-goods producing firms generate differentiated goods (indexed by z) using the technology $Y_t(z) = A_t H_t(z)$ where $Y_t(z)$ is output, A_t the productivity shock that all firms face, and $H_t(z)$ firm z's per-period hours input. The rest of the model, including aggregation across goods through the Dixit-Stiglitz apparatus, follows similarly standard lines (see e.g., King, 2000, or Woodford, 2003). We note, however, that relative price distortions mean that aggregate output is equal to $\frac{A_t}{\Delta_t}H_t$, $\Delta_t$being a per-period relative price distortion term, rather than $A_t H_t$; see Khan, King, and Wolman (2003). Following these authors, we set up the Lagrangian problem for aggregate welfare maximization as:

$$\pounds = \frac{C_t^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} - \frac{\chi_0 H_t^{1+\chi}}{1+\chi} +$$

$$\omega_{1t}\left[\frac{A_t}{\Delta_t}H_t - C_t - G_t\right] \tag{4}$$

$$-\omega_{2t}\left[\frac{A_t H_t}{\Delta_t C_t^{\sigma^{-1}}} + \alpha\beta E_t\left[\Pi_{t+1}^{\epsilon-1} Z_{1t+1}\right] - Z_{1t}\right] \tag{5}$$

$$-\omega_{3t}\left[\frac{\chi_0 H_t^{1+\chi}\nu_t}{(1-\tau)\left(1-\epsilon^{-1}\right)\Delta_t} + \alpha\beta E_t\left[\Pi_{t+1}^{\epsilon} Z_{2t+1}\right] - Z_{2t}\right] \tag{6}$$

$$+\omega_{4t}\left[(1-\alpha)\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{\epsilon}{\epsilon-1}} + \alpha\Pi_t^{\epsilon}\Delta_{t-1} - \Delta_t\right] \tag{7}$$

$$-\omega_{5t}\left[Z_{1t}\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{1}{1-\epsilon}} - Z_{2t}\right] \tag{8}$$

$$-\omega_{6t}\left[1-\beta E_t[(\frac{C_t}{C_{t+1}})^{-\frac{1}{\sigma}}\frac{1}{\Pi_{t+1}}]\right] \tag{9}$$

where νt represents exogenous variations in the desired markup. We have assumed separable preferences for the representative household, parameterized by $\frac{1}{\sigma}$ (the relative risk-aversion coefficient), $\frac{1}{\chi}$ (Frish labor supply elasticity), and β (the discount factor). C_t is the household's period t level of consumption and H_t is the representative household's total hours worked. Variables ω_{1t} to ω_{6t} respectively denote the Lagrange multipliers attached to the implementability constraints (4)-(9). Expression (4) is the economy's resource constraint.

It eases the analysis to cast firms' problem in recursive form. The recursive representation of profit maximization appears in constraints (5), (6), and (8). Π_t is the gross inflation rate for aggregate output, and the parameter ϵ is the demand elasticity for a typical firm's product. The relative price distortion, Δ_t, is represented by constraint (7) The household's Euler equation for consumption constitutes expression (9); reflecting the ZLB, this constraint incorporates a unit value for the gross nominal interest rate ($R_t = 1$).

Finally, A_t and G_t denote shocks to productivity and government purchases, respectively. They follow AR(1) processes:

$$\begin{aligned} \log A_t &= \rho_A \log A_{t-1} + \epsilon_{At} \\ \log G_t &= \rho_G \log G_{t-1} + \epsilon_{Gt} \end{aligned}$$

where ϵ_{At} and ϵ_{Gt} are all i.i.d. normal, mean zero innovations. These two exogenous shocks are sources of variation in the natural rate of interest, which becomes the trigger, in our experiments in Section 4, for the economy's reaching a zero lower bound for the nominal interest rate. Our derivations in this paper, however, pertain to perfect foresight dynamics of inflation and output gap, so that the $\{\epsilon_{At}\}$ and $\{\epsilon_{Gt}\}$ realizations that underly the natural-rate's trajectory are in period-t information sets.

First-Order Conditions for Optimal Policy We now characterize the nonlinear optimal policy problem. The first-order conditions with respect to consumption, hours worked, relative price distortion, gross goods inflation, and the variables Z_{1t} and Z_{2t}, are

$$C_t^{-\sigma} - \omega_{1t} + \sigma^{-1}(\frac{A_t H_t}{\Delta_t}) C_t^{-(1+\sigma^{-1})}\omega_{2t} - \sigma^{-1}\omega_{6t-1}\frac{(\frac{C_{t-1}}{C_t})^{\sigma^{-1}}}{C_t\Pi_t} + \sigma\beta\omega_{6t}E_t[\frac{(\frac{C_t}{C_{t+1}})^{\sigma^{-1}}}{C_t\Pi_{t+1}}] = 0.$$

$$-\chi_0 H_t^{\chi} + \omega_{1t}\frac{A_t}{\Delta_t} - \omega_{2t}\frac{A_t}{\Delta_t C_t^{\sigma^{-1}}} - \omega_{3t}\frac{\chi_0(1+\chi)H_t^{\chi}\nu_t}{(1-\tau)(1-\epsilon^{-1})\Delta_t} = 0,$$

$$\frac{A_t H_t \omega_{2t}}{\Delta_t^2 C_t^{\sigma}} + \frac{\chi_0 H_t^{1+\chi}\omega_{3t}\nu_t}{(1-\tau)\left(1-\epsilon^{-1}\right)\Delta_t^2} - \omega_{4t} + \alpha\beta E_t\left[\omega_{4t+1}\Pi_{t+1}^{\epsilon}\right] = \frac{A_t H_t \omega_{1t}}{\Delta_t^2}.$$

$$
\begin{aligned}
&-\alpha[(\epsilon-1)\omega_{2t-1}\Pi_t^{\epsilon-2}Z_{1t}+\epsilon\omega_{3t-1}\Pi_t^{\epsilon-1}Z_{2t}]\\
&-\alpha\epsilon\left(\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{1}{\epsilon-1}}-\Pi_t\Delta_{t-1}\right)\omega_{4t}\Pi_t^{\epsilon-2}-\frac{\alpha\Pi_t^{\epsilon-2}}{1-\alpha}\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{-\epsilon}{\epsilon-1}}Z_{1t}\omega_{5t}\\
&-\omega_{6t-1}\frac{(\frac{C_{t-1}}{C_t})^{\sigma^{-1}}}{\Pi_t^2}=0
\end{aligned}
$$

$$\omega_{2t}-\alpha\omega_{2t-1}\Pi_t^{\epsilon-1}-\omega_{5t}(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha})^{\frac{1}{1-\epsilon}}=0$$

$$\omega_{3t}-\alpha\omega_{3t-1}\Pi_t^{\epsilon}+\omega_{5t}=0$$

Loglinear Approximations to the First-Order Conditions We now make a combined loglinear/linear approximation to the Ramsey problem. We first note that a set of Lagrange multipliers take zero value in the nonstochastic steady state with zero inflation rate: $\omega_2=\omega_3$ $=\omega_5=\omega_6=0$. Notice that this implies that the desired markup shock, ν_t does not appear in the linearized version of the nonlinear first-order conditions. But because we want the approximation to the model to be faithful to the fact that these series do fluctuate around their mean, we take *linear approximations* for those expressions where the multipliers enter; and use loglinear approximations to the remaining variables in the model. Let $\hat{x}_t$ refer to log-deviations of the variable X_t from its steady-state level $\overline{X}$. We assume that the optimal monetary policy conditions on fiscal decisions, in the sense that the Ramsey policymaker takes government purchases as preordained, not as a choice variable. The conditions that appear in the approximated model are:

$$-\sigma^{-1}\hat{C}_t-\hat{\omega}_{1t}+\frac{\sigma^{-1}}{s_c}\omega_{2t}+\sigma^{-1}\beta C^{\sigma^{-1}-1}(\omega_{6t}-\beta^{-1}\omega_{6t-1})=0$$

$$-\chi\hat{H}_t+\hat{\omega}_{1t}+\hat{A}_t-\omega_{2t}-(1+\chi)\omega_{3t}=0$$

$$\frac{C^{1-\sigma^{-1}}}{\omega_4 s_c}\omega_{2t}-\frac{\chi_0H^{1+\chi}}{(1-\tau)(1-\epsilon^{-1})\omega_4}\omega_{3t}-\hat{\omega}_{4t}+\alpha\beta E_t[\hat{\omega}_{4t+1}+\epsilon\hat{\Pi}_{t+1}]=\hat{A}_t+\hat{H}_t+\hat{\omega}_{1t}$$

$$\hat{\Pi}_t=(\frac{1-\alpha}{\alpha\epsilon\omega_4})\omega_{6t-1}+\frac{1}{\epsilon}(\omega_{2t}-\omega_{2t-1})$$

$$\omega_{2t}=\alpha\omega_{2t-1}+\omega_{5t}$$

$$\omega_{3t}=\alpha\omega_{3t-1}-\omega_{5t},$$

s_c being the steady-state share of government purchases in total spending.

It can be established that the optimal inflation rate is unaffected by the presence of the productivity and public spending shocks. Proceeding toward this goal, let us almagamate the first two conditions given above:

$$\sigma^{-1}\hat{C}_t + \chi\hat{H}_t - \hat{A}_t + (1 - \frac{\sigma^{-1}}{s_c})\omega_{2t} + (1+\chi)\omega_{3t} - \sigma^{-1}\beta C^{\sigma^{-1}-1}(\omega_{6t} - \beta^{-1}\omega_{6t-1}) = 0$$

and write out explicitly the output gap identity, corresponding to logarithmic difference between equilibrium GDP under sticky prices with the (log-deviation for the) flexible-price output level:

$$x_t = \hat{Y}_t - \hat{Y}_t^*$$

We also have a number of "gap" relations among the model's loglinearized variables. The resource constraint and technology reveal a proportionality between output, consumption, and hours gaps:

$$x_t = s_c(\hat{C}_t - \hat{C}_t^*) = \hat{H}_t - \hat{H}_t^*$$

Labor hiring (implying equality of marginal product of labor and marginal rate of substitution) and the resource constraint imply that under flexible prices, we have

$$\sigma^{-1}\hat{C}_t^* + \chi\hat{H}_t^* = \hat{A}_t$$

In addition, loglinear expressions for flexible-price consumption and output are:

$$\hat{C}_t^* = -\frac{\chi}{(\sigma^{-1}/s_c)+\chi}\hat{g}_t + \frac{\chi+1}{(\sigma^{-1}/s_c)+\chi}\hat{A}_t; \quad \hat{Y}_t^* = \frac{(\sigma^{-1}/s_c)}{(\sigma^{-1}/s_c)+\chi}\hat{g}_t + \frac{\chi+1}{(\sigma/s_c)+\chi}\hat{A}_t$$

Deploying these relations, we can rewrite the first-order condition for output as:

$$((\sigma^{-1}/s_c)+\chi)x_t + (1-(\sigma^{-1}/s_c))\omega_{2t} + (1+\chi)\omega_{3t} - \sigma^{-1}\beta C^{\sigma^{-1}-1}(\omega_{6t} - \beta^{-1}\omega_{6t-1}) = 0.$$

Optimal conditions for ω_{2t} and ω_{3t} further imply:

$$\omega_{2t} + \omega_{3t} = \alpha(\omega_{2t-1} + \omega_{3t-1})$$

It is convenient to define $\omega_{7t} = \omega_{2t} + \omega_{3t}$ and its definition implies the law of motion,

$$\omega_{7t} = \alpha\omega_{7t-1.}$$

Steady-state values of ω_{7t} are zero. Therefore, the Ramsey policy that starts with period 0 conditions entails $\omega_{7t} = 0$, $t = 0, \cdots, \infty$. Taking this relation into account, and rearranging the preceding optimality conditions, we obtain:

$$x_t - \omega_{2t} - \frac{\sigma^{-1}\beta C^{\sigma^{-1}-1}}{(\sigma^{-1}/s_c) + \chi}(\omega_{6t} - \beta^{-1}\omega_{6t-1}) = 0.$$

A relation for inflation is implied among the conditions above:

$$\pi_t = (\frac{1-\alpha}{\alpha\epsilon\omega_4})\omega_{6t-1} - \frac{1}{\epsilon}(\omega_{2t} - \omega_{2t-1})$$

where π_t $(= \hat{\Pi}_t)$ is the log-change in the aggregate goods price level between period t and $t-1$. The productivity and fiscal shocks are absent from this pair of equations, which fully capture the policymaker's optimality conditions.

The policy rules for the output gap and inflation implied by Ramsey policy thus apply irrespective of the specification of the productivurt and government purchase shocks, and thus make no direct reference to these shocks.

Having characterized the first order approximation to optimal monetary policy, we now demonstrate that this representation is equivalent to the optimal policy problem using a linear-quadratic approach.

Equivalence of Loglinearized (Nonlinear) Ramsey Solution and Linear-Quadratic Approaches To characterize optimal policy in a linear-quadratic version of the problem, we first take note of the fact that the two constraints for optimal policy are the New Keynesian Phillips curve and the intertermporal IS equation:

$$\begin{aligned} \pi_t &= \beta\ E_t\{\pi_{t+1}\} + \kappa\ x_t && (10) \\ x_t &= E_t\{x_{t+1}\} - \sigma E_t\{i_t - \pi_{t+1}\} && (11) \end{aligned}$$

Following Rotemberg and Woodford (1997) and Woodford (2003), an approximation of household utility can be written as:

$$\sum_{t=0}^{\infty}\beta^t E_0[\lambda\frac{\pi_t^2}{2} + \frac{x_t^2}{2}], \qquad (12)$$

where we have normalized the weight on output-gap variability, leaving $\lambda = \epsilon/\kappa$ as the weight on inflation variability.

Let us define

$$\phi_{1t} = -\frac{\lambda\sigma^{-1}\beta C^{\sigma^{-1}-1}}{(\sigma^{-1}/s_c)+\chi}\omega_{6t}; \quad \phi_{2t} = \frac{\lambda}{\kappa}\omega_{2t}$$

where $\kappa = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha((\sigma^{-1}/s_c)+\chi)}$. Substituting these expressions into the IS and Phillips curves, we obtain:

$$\pi_t = (\frac{s_c\sigma}{\beta})\phi_{1t-1} - (\phi_{2t} - \phi_{2t-1})$$

$$\lambda x_t = \kappa\phi_{2t} - (\phi_{1t} - \beta^{-1}\phi_{1t-1}).$$

These correspond to the first-order conditions associated to the linear-quadratic problem previosly described, and ϕ_{1t} and ϕ_{1t} denote the Lagrange multipliers attached to equations (10) and (11), respectively.

Implementation of the Equivalence Result under Perfect Foresight: Piecewise Linear Appoximation Let us consider the solution procedure for the case where, in period zero, a negative shock to the natural real rate of interest strikes. In order to obtain a nonlinear solution of the Ramsey problem under these circumstances, we begin with a guess on the date at which optimal policy is able to launch from the ZLB. Denote this conjectured date T_N^*. For given T_N^*, we partition history into a ZLB phase and post-ZLB phase. Having solved for post-ZLB dynamics, we move on to ZLB-era dynamics. We compute Lagrange multiplier values in order to verify that the value of the Lagrange multiplier attached to the IS equation is in line with our conjectured T_N^* value. The linear/loglinear approximation is used in our analysis once the conjecture matches the actual ZLB departure date.