Bayesian Methods for Macroeconometrics

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Introduction I

- Linear rational expectations models are often used as local approximations to DSGE models. In many DSGE models the local equilibrium dynamics are not unique.

- Why do we care? *Indeterminacy* often arises in monetary DSGE models, for instance, if the central bank does not raise the nominal interest rate aggressively enough in response to inflation as to increase the real rate.

- Under *indeterminacy*, purely extrinsic *belief shocks* or *sunspot shocks* can influence equilibrium allocations and induce business cycle fluctuations that would not be present under *determinacy*.

- Hence, a central bank that wants to stabilize aggregate fluctuations should avoid policies that lead to *indeterminacy*. *Indeterminacy* thus becomes an issue for policy design as well as for our understanding of business cycles.

Simple Example (I)

- Simple model:

$$y_t = \frac{1}{\theta} I\!E_t[y_{t+1}] + \epsilon_t, \tag{1}$$

where $\epsilon_t \sim iid(0, 1)$ and $\theta \in \Theta = [0, 2]$.

- Introduce conditional expectation $\xi_t = I\!E_t[y_{t+1}]$ and forecast error $\eta_t = y_t - \xi_{t-1}$.

- Thus,

$$\xi_t = \theta\xi_{t-1} - \theta\epsilon_t + \theta\eta_t. \tag{2}$$

- Canonical LRE model (Sims, 2002)

$$\Gamma_0(\theta)s_t = \Gamma_1(\theta)s_{t-1} + \Psi(\theta)\epsilon_t + \Pi(\theta)\eta_t, \tag{3}$$

Simple Example (II)

- Determinacy: $\theta > 1$. Then only stable solution:

$$\xi_t = 0, \quad \eta_t = \epsilon_t, \quad y_t = \epsilon_t \tag{4}$$

- Indeterminacy: $\theta \leq 1$ the stability requirement imposes no restrictions on forecast error:

$$\eta_t = \widetilde{M}\epsilon_t + \zeta_t. \tag{5}$$

- For simplicity assume now $\zeta_t = 0$. Then

$$y_t - \theta y_{t-1} = \widetilde{M}\epsilon_t - \theta\epsilon_{t-1}. \tag{6}$$

- Reparameterization $\widetilde{M} = 1 + M$. The indeterminacy region of the parameter space is labelled $\Theta^I = [0, 1]$.

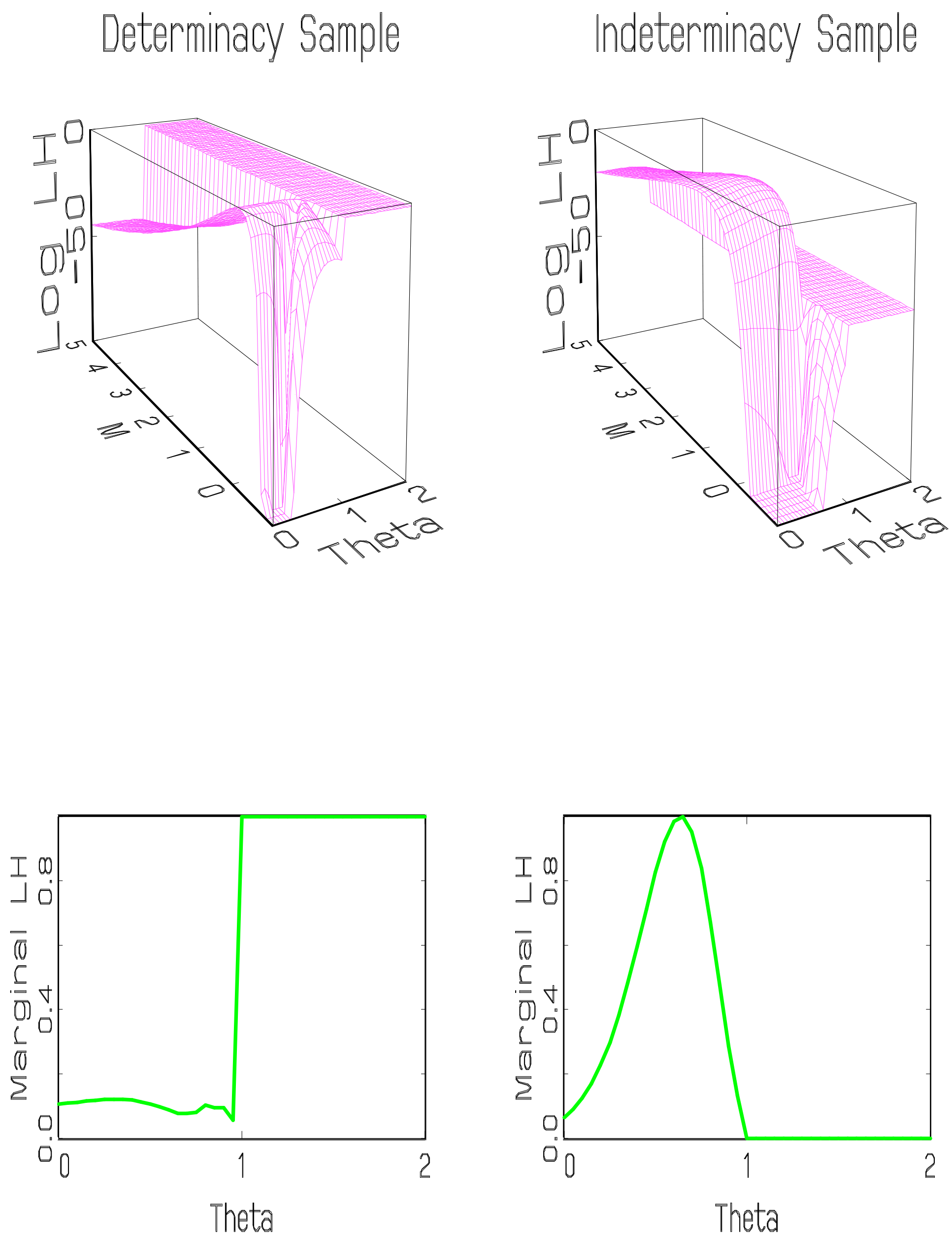


Figure 1: Likelihood / Posterior Density for Single Equation Model

Notes: First row can be interpreted as joint *log* posterior density of θ and M (under a uniform prior distribution) standardized by the posterior mode. Second row depicts the marginal posterior of density of θ, standardized by its mode.

Simple Example (II)

- Classical LR test is non-standard:

$$LR = \frac{\sup_{0 \leq \theta \leq 1, M} \ \mathcal{L}_I(\theta, M|Y^T)}{\mathcal{L}_D(Y^T)}. \tag{7}$$

- Andrews and Ploberger (1994)'s optimal test

$$LR_{ave} = \int \frac{\mathcal{L}_I(\theta, M|Y^T)}{\mathcal{L}_D(Y^T)} w(\theta, M) \ d\theta \cdot dM \tag{8}$$

Simple Example (III)

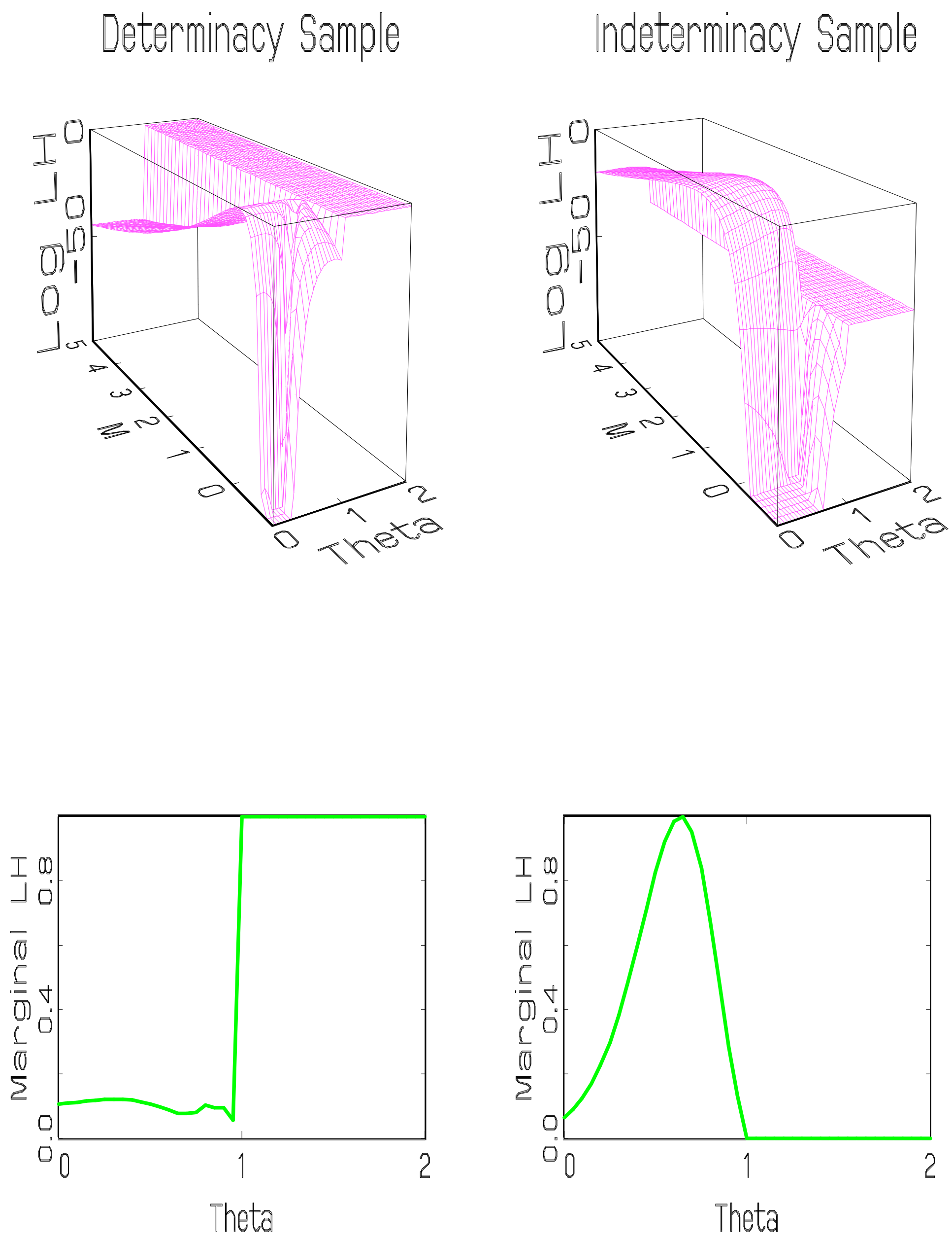
- Bayesian approach: place prior on θ, M.

- Posterior distribution:

$$p(\theta, M|Y^T) = \frac{(\{\theta \in \Theta^I\}\mathcal{L}_I(\theta, M|Y^T) + \{\theta \in \Theta^D\}\mathcal{L}_D(Y^T))p(\theta, M)}{\int \mathcal{L}(\theta, M|Y^T)p(\theta, M)d\theta \cdot dM}. \tag{9}$$

- Marginal posterior of θ:

$$p(\theta|Y^T) \propto \frac{1}{12}\{\theta \in \Theta^I\} \int \mathcal{L}_I(\theta, M|Y^T)dM + \frac{1}{2}\{\theta \in \Theta^D\}\mathcal{L}_D(Y^T), \tag{10}$$



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A Monetary Model (I)

- The following equations provide a log-linear approximation to an optimization-based representative agent model.
- Endogenous variables: real output $\widetilde{y}_t$, inflation $\widetilde{\pi}_t$, and the nominal interest rate $\widetilde{R}_t$.
- *Consumption Euler equation*:

$$\widetilde{y}_t = I\!E_t[\widetilde{y}_{t+1}] - \tau(\widetilde{R}_t - I\!E_t[\widetilde{\pi}_{t+1}]) + g_t, \tag{11}$$

- *Price setting equation*:

$$\widetilde{\pi}_t = \beta I\!E_t[\widetilde{\pi}_{t+1}] + \kappa\left[\widetilde{y}_t - z_t\right], \tag{12}$$

A Monetary Model (II)

- *Monetary policy reaction function*:

$$\widetilde{R}_t = \rho \widetilde{R}_{t-1} + (1-\rho_R)(\psi_1 \widetilde{\pi}_t + \psi_2[\widetilde{y}_t - z_t]) + \epsilon_{R,t}, \tag{13}$$

where ρ_R controls interest rate smoothing and the central bank targets current inflation and output with policy parameters ψ_2 and ψ_2 respectively.

- *Exogenous stochastic processes*:

$$\ln g_t = (1-\rho_g)\ln g + \rho_g \ln g_{t-1} + \epsilon_{g,t}, \tag{14}$$

$$\ln z_t = (1-\rho_z)\ln z + \rho_z \ln z_{t-1} + \epsilon_{z,t}. \tag{15}$$

and zero-mean innovations $\epsilon_{R,t}$, $\epsilon_{g,t}$ and $\epsilon_{z,t}$ serially uncorrelated.

- Sunspot shocks ζ_t, which may affect equilibrium allocations under *indeterminacy*.

Generalization (I)

- Canonical form:

$$\Gamma_0(\theta)s_t = \Gamma_1(\theta)s_{t-1} + \Psi(\theta)\epsilon_t + \Pi(\theta)\eta_t, \tag{16}$$

- The system can be rewritten as

$$s_t = \Gamma_1^*(\theta)s_{t-1} + \Psi^*(\theta)\epsilon_t + \Pi^*(\theta)\eta_t. \tag{17}$$

- Replace Γ_1^* by $J\Lambda J^{-1}$ and define $w_t = J^{-1}s_t$.

- To deal with repeated eigenvalues and non-singular Γ_0 we use Generalized Complex Schur Decomposition (QZ) in practice.

- Let the i'th element of w_t be $w_{i,t}$ and denote the i'th row of $J^{-1}\Pi^*$ and $J^{-1}\Psi^*$ by $[J^{-1}\Pi^*]_{i.}$ and $[J^{-1}\Psi^*]_{i.}$, respectively.

Generalization (II)

- Rewrite model:

$$w_{i,t} = \lambda_i w_{i,t-1} + [J^{-1}\Pi^*]_{i.}\epsilon_t + [J^{-1}\Psi^*]_{i.}\eta_t. \tag{18}$$

- Define the set of stable AR(1) processes as

$$I_s(\theta) = \left\{ i \in \{1, \ldots n\} \middle| \, |\lambda_i(\theta)| \le 1 \right\} \tag{19}$$

- Let $I_x(\theta)$ be its complement. Let Ψ_x^J and Π_x^J be the matrices composed of the row vectors $[J^{-1}\Psi^*]_{i.}$ and $[J^{-1}\Pi^*]_{i.}$ that correspond to unstable eigenvalues, i.e., $i \in I_x(\theta)$.

- Stability condition:

$$\Psi_x^J \epsilon_t + \Pi_x^J \eta_t = 0 \tag{20}$$

for all t.

Generalization (III)

- Solving for η_t. Define

$$\Pi_x^J = \begin{bmatrix} U_{.1} & U_{.2} \end{bmatrix} \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{.1}' \\ V_{.2}' \end{bmatrix} = \underbrace{U}_{m\times m} \underbrace{D}_{m\times k} \underbrace{V'}_{k\times k} = \underbrace{U_{.1}}_{m\times r} \underbrace{D_{11}}_{r\times r} \underbrace{V_{.1}'}_{r\times k}, \tag{21}$$

Proposition 1 *If there exists a solution to Eq. (20) that expresses the forecast errors as function of the fundamental shocks ϵ_t and sunspot shocks ζ_t, it is of the form*

$$\begin{aligned} \eta_t &= \eta_1 \epsilon_t + \eta_2 \zeta_t \qquad (22) \\ &= (-V_{.1} D_{11}^{-1} U_{.1}' \Psi_x^J + V_{.2} \widetilde{M}) \epsilon_t + V_{.2} M_\zeta \zeta_t, \end{aligned}$$

where $\widetilde{M}$ is an $(k-r) \times l$ matrix, M_ζ is a $(k-r) \times p$ matrix, and the dimension of $V_{.2}$ is $k \times (k-r)$. The solution is unique if $k = r$ and $V_{.2}$ is zero.

Generalization (IV)

- Law of motion for s_t:

$$s_t = \Gamma_1^*(\theta)s_{t-1} + [\Psi^*(\theta) - \Pi^*(\theta)V_{.1}(\theta)D_{11}^{-1}(\theta)U_{.1}'(\theta)\Psi_x^J(\theta)]\epsilon_t \qquad (23)$$

$$+\Pi^*(\theta)V_{.2}(\theta)(\widetilde{M}\epsilon_t + M_\zeta\zeta_t).$$

- Reparameterization:

$$\widetilde{M} = M^*(\theta) + M \qquad (24)$$

Generalization (V)

- Constructing a benchmark indeterminacy solution.

- For every vector $\theta \in \Theta^I$ we construct a vector $\tilde{\theta} = g(\theta)$ that lies on the boundary of the determinacy region.

- Compare

$$\frac{\partial s_t}{\partial \epsilon_t'}(\theta, M) = \Psi^*(\theta) - \Pi^*(\theta)V_{.1}(\theta)D_{11}^{-1}(\theta)U_{.1}'(\theta)\Psi_x^J(\theta) + \Pi^*(\theta)V_{.2}(\theta)\widetilde{M} \tag{25}$$

$$= B_1(\theta) + B_2(\theta)\widetilde{M}$$

to

$$\frac{\partial s_t}{\partial \epsilon_t'}(g(\theta), \cdot) = B_1(g(\theta)). \tag{26}$$

- Projection:

$$M^*(\theta) = [B_2(\theta)' B_2(\theta)]^{-1} B_2(\theta)' * [B_1(g(\theta)) - B_1(\theta)] \tag{27}$$

Discussion

- We consider all indeterminacy solutions - normalization to center the prior.
- Determinacy region might depend on all model parameters.
- Information about indeterminacy: number of autoregressive roots, parameter restrictions.
- Full-information approach: more efficient but potentially subject to misspecification (omitted dynamics).
- Limited-information approach such as GMM: less efficient and subject to hidden identification problems.

  If the exogenous shocks in the New Keynesian model are serially uncorrelated then ψ_1 cannot be identified in a Taylor rule regression under determinacy.