

Bayesian Analysis of DSGE Models

Frank Schorfheide

Department of Economics, University of Pennsylvania

Likelihood Function (I)

- Forecast error covariance for vectors of macro variables is typically not rank-deficient.
- In order to explain data, we need a model that delivers a non-degenerate probability distribution for the observables:
 - Include enough structural shocks: e.g. preference shocks. Capture to some extent aggregation effects.
 - Include measurement errors. More precisely, these errors are supposed to capture the discrepancy between model and reality.
- Generalization of shock structure potentially breaks cross-coefficient restrictions. Might reduce misspecification but also introduce additional identification problems.
- Endogenous propagation of innovations versus exogenous propagation.

Likelihood Function (II)

- Likelihood function: joint probability density of the data $Y^T = \{y_1, \dots, y_t\}$:

$$\mathcal{L}(\theta|Y^T) = p(Y^T|\theta) = \prod_{t=1}^T p(y_t|Y^{t-1}, \theta) \quad (1)$$

- Log-linearized DSGE models can be written as state-space models:

$$\text{measurement} : y_t = A(\theta) + B(\theta)s_t \quad (2)$$

$$\text{state transition} : s_t = \Phi_1(\theta)s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t. \quad (3)$$

- Make distributional assumption: $\epsilon_t \sim iid\mathcal{N}(0, \Sigma_\epsilon(\theta))$.
- It is only assumed that the y_t 's are observable. The vector s_t may have unobservable elements such as conditional expectations or a latent productivity process.
- We obtained the state transition equation when we solved the LRE model.

Likelihood Function (III)

- If s_t is not fully observable we need to use a filter to obtain the likelihood function.

This filter is a recursive algorithm to calculate

$$p(y_t|Y^{-1}, \theta), \quad t = 1, \dots, T$$

- Iterations:

- Initialization at time t : $p(s_t|Y^t, \theta)$

- Forecasting $t + 1$ given t :

1. Transition equation: $p(s_{t+1}|Y^t, \theta) = \int p(s_{t+1}|s_t, Y^t, \theta)p(s_t|Y^t, \theta)ds_t$

2. Measurement equation: $p(y_{t+1}|Y^t, \theta) = \int p(y_{t+1}|s_{t+1}, Y^t, \theta)p(s_{t+1}|Y^t, \theta)ds_{t+1}$

- Updating with Bayes theorem. Once y_{t+1} becomes available:

$$p(s_{t+1}|Y^{t+1}, \theta) = p(s_{t+1}|y_{t+1}, Y^t, \theta) = \frac{p(y_{t+1}|s_{t+1}, Y^t, \theta)p(s_{t+1}|Y^t, \theta)}{p(y_{t+1}|Y^t, \theta)}$$

Likelihood Function (IV)

- Initialization: process s_t is stationary we can initialize the filter with the unconditional distribution of s_t , calculated from the transition equation.

$$\mathbf{E}[s_t s_t'] = \Phi_1 \mathbf{E}[s_t s_t'] \Phi_1' + \Phi_\epsilon \Sigma_\epsilon \Phi_\epsilon'$$

- Iterations look difficult because they involve integrations. But:
- If ϵ_t is normally distributed, all conditional distributions are also normal.
- At each step we only track means and covariance matrices... Kalman filter.
- Kalman filter iterations can be found in standard time series textbooks, e.g., Hamilton (1994). (see Appendix).

Extensions – Conditioning (I)

- Conditioning: so far we discussed the unconditional likelihood function.
- Other time series models, such as VARs, are commonly estimated based on conditional likelihood functions, e.g.,

$$p(y_1, \dots, y_T | y_0, \theta) \tag{4}$$

- In order to obtain a conditional likelihood for the DSGE, note that

$$p(y_1, \dots, y_T | y_0, \theta) = \frac{p(y_0, y_1, \dots, y_T | \theta)}{p(y_0 | \theta)} \tag{5}$$

- The numerator can be obtained by applying the (Kalman) filter to all observations, including y_0 , whereas the denominator is obtained by applying the filter to y_0 only.

Extensions – Conditioning (II)

- Short-cut: the Kalman filter applied to y_0, \dots, y_T generates $p(y_t|Y^T)$, $t = 0, \dots, T$.

Define likelihood as

$$\mathcal{L} = \prod_{t=1}^T p(y_t|Y^T)$$

Extensions – Nonstat Levels (I)

- Reference: Chang, Doh, Schorfheide (2005): “Non-stationary Hours in a DSGE Model.”
- Log-linearized DSGE has state space representation of the form:

$$y_t = \Gamma_0 + \Gamma_1 s_{1,t} + \Gamma_2 s_{2,t} + \Gamma_3 t \quad (6)$$

$$s_{1,t} = \Phi_1 s_{1,t-1} + \Psi_1 \epsilon_t \quad (7)$$

$$s_{2,t} = s_{2,t-1} + \Psi_2 \epsilon_t. \quad (8)$$

- Trend in (6) captures drift random walk technology A_t .
- Equation (7) represents the law of motion for the variables of the detrended model.
- (8) describes evolution of $s_{2,t} = \ln A_t - \gamma t$ for \mathcal{M}_0 and $s_{2,t} = [\ln A_t - \gamma t, \ln B_t]'$.

Extensions – Nonstat Levels (II)

- Initialization of Kalman filter: factorize the initial distribution as $p(s_{1,0})p(s_{2,0})$.
- Set the first component equal to the unconditional distribution of $s_{1,t}$.
- Absorb second component, composed of the distribution of $\ln A_0$ (\mathcal{M}_0) and $[\ln A_0, \ln B_0]'$ (\mathcal{M}_1), respectively, into the specification of our prior $p(\theta)$.
- If DSGE model implies, say, common trend in investment, consumption, and output, then estimation based on likelihood function in levels, incorporates information about the ratios investment/output, consumption/output.
- Roughly, likelihood will relate the model implied steady-state ratios to long-run averages in the data.

Likelihood Function (V)

- If DSGE models is solved with nonlinear method then Kalman Filter is not sufficient.
- Alternative: particle filter or sequential Monte Carlo filter.
- Particle has been used to analyze stochastic volatility models: Pitt and Shephard (1999), and Kim, Shephard, and Chib (1998);
- DSGE models solved with finite elements method: Fernandez-Villaverde and Rubio-Ramirez (2004).
- DSGE models solved with second-order perturbation methods: An (2005) and An and Schorfheide (2005).