Bayesian Methods for Macroeconometrics

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Some Theoretical Properties of VARs

- A vector autoregression is a generalization of the AR(p) model to the multivariate case:

$$y_t = \Phi_0 + \Phi_1 y_{t-1} + \ldots + \Phi_p y_{t-p} + u_t \tag{1}$$

The random variable y_t is a $n \times 1$ random vector that takes values in $I\!R^n$.

- For a theoretical analysis, it is often convenient to express the VAR(p) in the so-called companion form.

$$\begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \begin{bmatrix} \Phi_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_{p-1} & \Phi_p \\ I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p} \end{bmatrix} + \begin{bmatrix} u_t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2}$$

- Let $\xi_t = [y_t', y_{t-1}', \ldots, y_{t-p+1}']'$. The VAR can be rewritten as

$$\xi_t = F_0 + F_1 \xi_{t-1} + \nu_t. \tag{3}$$

Some Theoretical Properties of VARs

- Define the $n \times np$ matrix $M_n = [I, 0]$ where I is an $n \times n$ identity matrix.

- It can be easily verified that $y_t = M_n \xi_t$.

- For a vector autoregression to be covariance stationary it is necessary that all eigenvalues of the matrix F_1 are less than one in absolute value.

- The expected value of y_t has to satisfy the vector difference equation

$$I\!E[y_t] = \Phi_0 + \Phi_1 I\!E[y_{t-1}] + \ldots \Phi_p I\!E[y_{t-p}] \quad \text{for all } t \tag{4}$$

- If the eigenvalues of F_1 are all less than one in absolute values and the VAR was initialized in the infinite past, then the expected value is given by

$$I\!E[y_t] = [I - \Phi_1 - \ldots \Phi_t]^{-1} \Phi_0 \tag{5}$$

Some Theoretical Properties of VARs

- To calculate the autocovariances we will assume that $\Phi_0 = 0$. Consider the companion form

$$\xi_t = F_1 \xi_{t-1} + \nu_t \tag{6}$$

If the eigenvalues of F_1 are all less than one in absolute value and the VAR was initialized in the infinite past, than the autocovariance matrix of order zero has to satisfy the equation

$$\Gamma_{\xi\xi,0} = I\!\!E[\xi_t \xi_t'] = F_1 \Gamma_{\xi\xi,0} F_1' + I\!\!E[\nu_t \nu_t'] \tag{7}$$

Some Theoretical Properties of VARs

- **Definition:** Let A and B be 2×2 matrices with the elements
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$
  The *vec* operator is defined as the operator that stacks the columns of a matrix, that is,
$$vec(A) = [a_{11}, a_{21}, a_{12}, a_{22}]'$$
  and the Kronecker product is defined as
$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} \quad \Box$$

- **Lemma** Let A, B, C be matrices whose dimension are such that the product ABC exists. Then
$$vec(ABC) = (C' \otimes A)vec(B) \quad \Box$$

Some Theoretical Properties of VARs

- A closed form solution for the elements of the covariance matrix of ξ_t can be obtained as follows

$$\begin{aligned} vec(\Gamma_{\xi\xi,0}) &= (F_1 \otimes F_1) vec(\Gamma_{\xi\xi,0}) + vec(I\!\!E[\nu_t \nu_t']) \\ &= [I - (F_1 \otimes F_1)]^{-1} vec(I\!\!E[\nu_t \nu_t']) \qquad (8) \end{aligned}$$

Since

$$I\!\!E[\xi_t \xi_{t-h}'] = F I\!\!E[\xi_{t-1}\xi_{t-h}'] + I\!\!E[\nu_t \xi_{t-h}'] \qquad (9)$$

we can deduce that

$$\Gamma_{\xi\xi,h} = F_1^h \Gamma_{\xi\xi,0} \qquad (10)$$

- Notice that $\Gamma_{\xi\xi,-h} = \Gamma'_{\xi\xi,-h}$.

Some Theoretical Properties of VARs

- Once we have calculate that autocovariances for the companion form process ξ_t it is straightforward to obtain the autocovariances for the y_t process. Since $y_t = M_n \xi_t$ it follows that

$$\Gamma_{yy,h} = I\!\!E[y_t y_{t-h}'] = I\!\!E[M_n \xi_t \xi_{t-h}' M_n'] = M_n \Gamma_{\xi\xi,h} M_n' \qquad (11)$$

Some Theoretical Properties of VARs

- **Result**: Consider the vector autoregression

$$y_t = \Phi_0 + \Phi_1 y_{t-1} + \ldots + \Phi_p y_{t-p} + u_t$$

where $u_t \sim iid\mathcal{N}(0, \Sigma_u)$ with companion form

$$\xi_t = F_0 + F_1 \xi_{t-1} + \nu_t$$

Suppose that the eigenvalues of F_1 are all less than one in absolute values and that the vector autoregression was initialized in the infinite past.

$$I\!\!E[y_t] = [I - \Phi_1 - \ldots \Phi_t]^{-1} \Phi_0, \quad \Gamma_{yy,h} = M_n \Gamma_{\xi\xi,h} M_n' \quad \forall h$$

where

$$vec(\Gamma_{\xi\xi,0}) = [I - (F_1 \otimes F_1)]^{-1} vec(I\!\!E[\nu_t \nu_t']), \quad \Gamma_{\xi\xi,h} = F_1^h \Gamma_{\xi\xi,0} \quad h > 0.$$

Likelihood Function

- We will now derive the likelihood function for a Gaussian VAR(p), conditional on initial observations $y_0, \ldots, y_{-p+1}$.
- The likelihood function can be used for both Bayesian and frequentist inference.
- The density of y_t conditional on $y_{t-1}, y_{t-2}, \ldots$ and the coefficient matrices $\Phi_0, \Phi_1, \ldots, \Sigma$ is of the form

$$\begin{aligned} p(y_t|Y^{t-1}, \Phi_0, \ldots, \Sigma) \;\propto\; & |\Sigma|^{-1/2} \exp\left\{ -\frac{1}{2}(y_t - \Phi_0 - \Phi_1 y_{t-1} - \ldots - \Phi_p y_{t-p})' \right. \\ & \left. \times \Sigma^{-1}(y_t - \Phi_0 - \Phi_1 y_{t-1} - \ldots - \Phi_p y_{t-p}) \right\} \qquad (12) \end{aligned}$$

Likelihood Function

- Define the $(np+1) \times 1$ vector x_t as

$$x_t = [1, y'_{t-1}, \ldots, y'_{t-p}]'$$

- Moreover, define the matrixes

$$Y = \left[\begin{array}{c} y'_1 \\ \vdots \\ y'_T \end{array} \right], \quad X = \left[\begin{array}{c} x'_1 \\ \vdots \\ x'_T \end{array} \right], \quad \Phi = [\Phi_0, \Phi_1, \ldots, \Phi_p]'$$

- The conditional density of y_t can be written in more compact notation as

$$p(y_t|Y^{t-1}, \Phi, \Sigma) \propto |\Sigma|^{-1/2} \exp\left\{ -\frac{1}{2}(y'_t - x'_t\Phi)\Sigma^{-1}(y'_t - x'_t\Phi)' \right\} \qquad (13)$$

To manipulate the density we will use some matrix algebra facts.

Likelihood Function

- **Facts**:

  (i) Let a be a $n \times 1$ vector, B be a symmetric positive definite $n \times n$ matrix, and tr the trace operator that sums the diagonal elements of a matrix. Then

$$a'Ba = tr[Baa']$$

  (ii) Let A and B be two $n \times n$ matrices, then

$$tr[A + B] = tr[A] + tr[B]$$

Likelihood Function

- In a first step, we will replace the inner product in the expression for the conditional density by the trace of the outer product

$$p(y_t|Y^{t-1},\Phi,\Sigma) \propto |\Sigma|^{-1/2}\exp\left\{-\frac{1}{2}tr[\Sigma^{-1}(y_t'-x_t'\Phi)'(y_t'-x_t'\Phi)]\right\} \qquad (14)$$

- In the second step, we will take the product of the conditional densities of $y_1,\ldots,y_T$ to obtain the joint density. Let Y_0 be a vector with initial observations

$$\begin{aligned} p(Y|\Phi,\Sigma,Y_0) &= \prod_{t=1}^{T} p(y_t|Y^{t-1},Y_0,\Phi,\Sigma) \\ &\propto |\Sigma|^{-T/2}\exp\left\{-\frac{1}{2}\sum_{t=1}^{T} tr[\Sigma^{-1}(y_t'-x_t'\Phi)'(y_t'-x_t'\Phi)]\right\} \\ &\propto |\Sigma|^{-T/2}\exp\left\{-\frac{1}{2}tr\left[\Sigma^{-1}\sum_{t=1}^{T}(y_t'-x_t'\Phi)'(y_t'-x_t'\Phi)\right]\right\} \\ &\propto |\Sigma|^{-T/2}\exp\left\{-\frac{1}{2}tr[\Sigma^{-1}(Y-X\Phi)'(Y-X\Phi)]\right\} \qquad (15) \end{aligned}$$

Likelihood Function

- Define the “OLS” estimator

$$\hat{\Phi} = (X'X)^{-1}X'Y \tag{16}$$

and the sum of squared OLS residual matrix

$$S = (Y - X\hat{\Phi})'(Y - X\hat{\Phi}) \tag{17}$$

- It can be verified that

$$(Y - X\Phi)'(Y - X\Phi) = S + (\Phi - \hat{\Phi})'X'X(\Phi - \hat{\Phi}) \tag{18}$$

- This leads to the following representation of the likelihood function

$$p(Y|\Phi, \Sigma, Y_0) \propto |\Sigma|^{-T/2} \exp\left\{-\frac{1}{2}tr[\Sigma^{-1}S]\right\} \exp\left\{-\frac{1}{2}tr[\Sigma^{-1}(\Phi - \hat{\Phi})'X'X(\Phi - \hat{\Phi})]\right\} \tag{19}$$

Likelihood Function

- Let $\beta = vec(\Phi)$ and $\hat{\beta} = vec(\hat{\Phi})$. It can be verified that

$$tr[\Sigma^{-1}(\Phi - \hat{\Phi})' X' X (\Phi - \hat{\Phi})] = (\beta - \hat{\beta})' [\Sigma \otimes (X'X)^{-1}]^{-1} (\beta - \hat{\beta}) \tag{20}$$

- and the likelihood function has the alternative representation

$$p(Y|\Phi, \Sigma, Y_0) \propto |\Sigma|^{-T/2} \exp\left\{ -\frac{1}{2} tr[\Sigma^{-1} S] \right\} \exp\left\{ -\frac{1}{2} (\beta - \hat{\beta})' [\Sigma \otimes (X'X)^{-1}]^{-1} (\beta - \hat{\beta}) \right\} \tag{21}$$