Exercises

1. Compute the gain of the Kuznets Filter.

2. Download the file us_realgdp.xls. This file contains quarterly observations for real GDP for the U.S. from 1955:I to 2004:IV.

   a. Estimate an AR(4) model (including a constant) for y_t. Using this estimated model constuct and plot the spectrum for y.

   b. Construct the business cycle component for the logarithm real GDP using the following steps:

      i. Calculate the filter weights for a band-pass filter that passes frequencies corresponding to periods between 6 and 32 quarters. Truncate the filter after 48 leads and lags. Call this filter $d(L) = \sum_{i=-48}^{48} d_{|i|} L^i$. Plot the values of d_i as a function of i,

      ii. Use the estimated AR(4) model to construct forecasts of 48 future values of y_t and 48 backcasts of presample values of y. (To construct the backcasts note that covariance stationarity implies that you can "reverse" the order of the data and construct "forecasts" of this reversed time series.). Convert these forecasts/backcasts of y_t into forecasts and backcasts of $x_t = \ln(GDP_t)$. At the end of this process you will have a data series on x_t from 1949:1 - 2015:2. The data from 1960:1-2003:2 are the actual data, and the other values are the forecasts and backcasts.

      iii. Apply the 48-term bandpass filter to the "padded-out" x_t series. Plot the resulting series for 1960:1 $< t <$ 2004:4.

3. Below you will find a description of Census X-11/X-12, a commonly used seasonal adjustment filter. Calculate and plot the gain of this filter. From the plot, explain why this filter is used to seasonally adjust time series.

Linear Approximation to Census X-11 and X-12

In its additive form, X-11 is composed of a set of linear filters modified by a few nonlinear features and regression adjustments for holiday and trading day effects. Here I describe the linear features of X-11 for monthly data. It is convenient to do this in the additive model:

$$x_t = TC_t + S_t + I_t$$

where x denotes the observed data, and TC, S, and I denote the unobserved trend-cycle, seasonal and irregular components. (This ignores holiday and trading-day effects that are incorporated in X-11/X-12.) In what follows, let $SI_t = S_t + I_t$. The following summary of the linear features of X-11 is taken from Young (1968) and Wallis (1974).

The linear operations in X-11 can be summarized by the filter

$$x_t^{sa} = X11(L)x_t$$

where $X11(L)$ is a two-sided linear filter constructed in 8 steps:

X11-1. Form an initial estimate of TC as $\widehat{TC}_t^1 = A_1(L)x_t$, where $A_1(L)$ is the centered 12-month moving average filter $A_1(L) = \sum_{j=-6}^{6} b_j L^j$, with $b_{|6|} = 1/24$, and $b_j = 1/12$, for $-5 \le j \le 5$.

X11-2. Form an initial estimate of $S+I$ as $\widehat{SI}_t^1 = x_t - \widehat{TC}_t^1$

X11-3. Form an initial estimate of S_t as $\hat{S}_t^1 = S_1(L^{12})\widehat{SI}_t^1$, where $S_1(L^{12}) = \sum_{j=-2}^{2} c_j L^{12j}$, and where c_j are weights from a 3×3 moving average (*i.e.*, $1/9, 2/9, 3/9, 2/9, 1/9$).

X11-4. Adjust the estimates of S so that they add to zero (approximately) over any 12 month period as $\hat{S}_t^2 = S_2(L)\hat{S}_t^1$, where $S_2(L) = 1 - A_1(L)$, where $A_1(L)$ is defined in step 1.

X11-5. Form a second estimate of TC as $\widehat{TC}_t^2 = A_2(L)(x_t^1 - \hat{S}_t^2)$, where $A_2(L)$ denotes a "Henderson" moving average filter. (The 13-term Henderson moving average filter is given by $A_2(L)=\sum_{i=-6}^{6} A_{2,i}L^i$, with $A_{2,0}=.2402$, $A_{2,|1|}=.2143$, $A_{2,|2|}=.1474$, $A_{2,|3|}=.0655$, $A_{2,|4|}=0$, $A_{2,|5|}=-.0279$, $A_{2,|6|}=-.0194$.)

X11-6. Form a third estimate of S as $\hat{S}_t^3=S_3(L^{12})(x_t-\widehat{TC}_t^2)$, where $S_3(L^{12})=\sum_{j=-3}^{3} d_j L^{12j}$, and where d_j are weights from a 3×5 moving average (*i.e.*, $1/15$, $2/15$, $3/15$, $3/15$, $3/15$, $2/15$, $1/15$).

X11-7. Adjust the estimates of S so that they add to zero (approximately) over any 12 month period as $\hat{S}_t^4=S_2(L)\hat{S}_t^3$, where $S_2(L)$ is defined in step 4.

X11-8. Form a final seasonally adjusted value as $x_t^{sa}=x_t-\hat{S}_t^4$.

4. Data: Attached you will find an Excel file that contains data on quarterly values of U.S. real GDP from 1955:1 through 2004:4.

Background: There has been considerable recent research on the apparent decline in volatility of real economic activity in the United States. Several researchers have dated the time of the volatility decline to be sometime near 1984. In this exercise you will use data on U.S. real GDP to study this decline in volatility. (If you are interested in reading some of the recent literature on this subject, see the survey paper "Has the Business Cycle Changed and Why?" by Stock and Watson, NBER Macro Annual 2002)

Let $x_t = 400\times\ln(\text{GPD}_t / \text{GDP}_{t-1})$ denote the quarterly rate of growth of U.S. real GDP expressed as an annual percent.

a. Plot the series from 1956:1 through 2004:4. Is the reduction in volatility evident?

b. Estimate an AR(1) model for x_t over 1956-1983 and 1984-2004.

(i) Compare the point estimates for the constant, AR coefficient and standard error of the regression. What seems to have changed across the two sample periods?

(ii) Test the hypothesis that the AR coefficient is the same in both periods.

(iii) Test the hypothesis that the constant term is the same over both periods.

(iv) Test the joint hypothesis that the AR coefficient and constant are the same over both periods.

c. Ahmed, Levin and Wilson (researchers at the Federal Reserve) estimate the spectrum of GDP growth over the two subsamples. They argue that the shape of spectrum is essentially the same in both periods, but that it's scale has changed. Use the estimated AR models in c to compute estimated spectra of GDP growth rates in each of the two samples. Plot the estimated spectra. Has the shape changed noticeably. What about the scale? Do you agree with Ahmed, Levin and Wilson's characterization?

d. I want you to test for a shift it the variance of ε_t in 1984. You will do this using procedures like those in the "Testing for Instability in Regression Models" portion of the class notes. First, I want to modify the analysis in the notes so that it applies to this problem. Suppose that ε_t is an iid sequence with $E(\varepsilon_t) = 0$, $E(\varepsilon_t^2) = \sigma_\varepsilon^2$ and $E(\varepsilon_t^4) = \kappa < \infty$. Let $y_t = \varepsilon_t^2$. Write $y_t = \beta + a_t$, where $\beta = E(y_t)$.

(i) Show that $\beta = \sigma_\varepsilon^2$. Show that a_t is i.i.d with mean 0 and variance σ_a^2. Show that $\sigma_a^2 < \infty$.

(ii) Let $\hat{\beta}_1$ denote the least squares estimator of β constructed using the first τ observations (so that $\hat{\beta}_1 = \frac{1}{\tau}\sum_{t=1}^{\tau} y_t$). Show that $\sqrt{\tau}(\hat{\beta}_1 - \beta) \xrightarrow{d} N(0, V_1)$. Derive an expression for V_1. How would you estimate V_1? Use this limiting result to motivate an approximation $\hat{\beta}_1 \overset{a}{\sim} N(\beta, \frac{1}{\tau}\hat{V}_1)$, where $\hat{V}_1$ is your estimator of V_1.

(iii) Repeat part (a) for $\hat{\beta}_2$, the least squares estimator of β constructed using observations $\tau + 1$ through T.

(iv). Let

$$\xi_W(\tau) = \frac{(\hat{\beta}_1 - \hat{\beta}_2)^2}{[\frac{1}{\tau}\hat{V}_1 + \frac{1}{T-\tau}\hat{V}_2]}$$

denote a test statistic for testing that the mean of y (equivalently the variance of ε) is constant across the two sample periods. Show that $\xi_W(\tau)$ is distributed as a χ_1^2 in large samples.

e. Estimate an AR(1) model over the sample period 1956-2001 and let $\hat{\varepsilon}_t$ denote the residuals from this regression. Let τ = 1983:4 and carry out the test that you derived in (d) using the OLS residuals in place of ε_t. (It turns out that asymptotic approximations that you derived in (5c) and also valid for the OLS residuals.) Is there evidence of a change in variance? Explain.

f. A problem with the analysis in (e) is that it imposes a "break date" of 1984, and this date was determined after looking at the data. Using the OLS residuals from 6, carry out a test for shift in the variance at an unknown break date using the QLR procedure over the middle 70% of the sample. Is there still evidence of a break. Looking at the individual $\xi_W(\tau)$ statistics over the different dates, where does the break seem to have occurred?

g. An alternative to the discrete break in the variance is that it evolved slowly over the period. This slow evolution can be captured using a GARCH(1,1) model. (NOTE .. WE HAVE COVERED THESE MODELS, BUT YOU MAY KNOW THEM AND MAY FIND THE RESULTS INTERESTING.)

(i) Using the OLS residuals from 6, estimate a GARCH(1,1) model for the variance function. Is there evidence of time variation?

(ii) Plot the estimated standard deviations. Does it look like there was discrete shift in the early 1980's? Does it look like a declining trend? Explain.

5. Attached you will find an Excel file PCE.XLS that contains quarterly values of the PCE price deflator for the United States from 1959:1-2004:4. Call these observations P_t. Let $p_t = \ln(P_t)$ denote the logarithm of the price index, $\pi_t = 400(p_t - p_{t-1})$ denote quarterly observations on the rate of price inflation (expressed in percentage points at an annual rate), and $\Delta\pi_t = \pi_t - \pi_{t-1}$ denote the change in the inflation rate.

a. Using the sample period 1961:1-2004:4.

i. Estimate an MA(1) for $\Delta\pi_t$.

ii. Estimate an MA(2) model for $\Delta\pi_t$. Is the second lag significant? Are the results consistent with the contention that $\Delta\pi_t$ is well described by the MA(1) model estimated in part (i)?

b. Suppose that $\pi_t = \tau_t + \varepsilon_t$, where $\tau_t = \tau_{t-1} + e_t$, and where $\{\varepsilon_t\}$ and $\{e_t\}$ are mutually uncorrelated white noise processes with variances σ_ε^2 and σ_e^2. Show that $\Delta\pi_t$ can be described by the MA(1) process $\Delta\pi_t = (1-\theta L)a_t$. Derive an expression for θ and σ_a^2 in terms of σ_ε^2 and σ_e^2.

c. Using the estimated MA(1) model in (1), construct estimates of σ_ε^2 and σ_e^2.

d. For any variable x_t, let $x_{t+k/t} = E[x_{t+k} \mid \{\pi_i\}_{i=1}^t]$.

i. Using the (τ, ε) representation of π, show that (i) $\tau_{t+1/t} = \tau_{t/t}$, (ii) $\pi_{t+1/t} = \tau_{t+1/t}$, and therefore (iii) $\pi_{t+1/t} = \tau_{t/t}$.

ii. Using the MA(1) representation for $\Delta\pi_t$ show that $\pi_{t+1/t} = \pi_t - \theta a_t$.

iii. Because the (τ, ε) and MA(1) representations are equivalent representations of the π process (as you showed in 2), use (a) and (b) to conclude that $\tau_{t/t} = \pi_t - \theta a_t$.

iv. Use the estimated values of σ_ε^2 and σ_e^2 from 4b and the Kalman filter to compute $\tau_{t/t}$ for $t = 1961{:}1\text{-}2004{:}4$. Use the residuals from the MA(1) model in 3 to compute $\pi_t - \hat{\theta}\hat{a}_t$ for $t = 1961{:}1\text{-}2004{:}4$. Does your Kalman filter estimate equal $\pi_t - \hat{\theta}\hat{a}_t$? Explain why or why not.

v. Repeat questions a(i) and c using the sample period 1961:1-1983:4.

vi. Repeat questions a(i) and c using the sample period 1984:1-2004:4.

vii. Have there been important changes in the inflation process? Are the changes statistically significant? Explain.

viii. Plot the implied spectrum of $\Delta\pi$ for the two sample periods. How has it changed?

5. Compute the 10th, 25th, 50th, 75th, and 95th percentiles of the distribution of $\int_0^1 W(s)^2 ds$.

6. Attached you will find an Excel file, TB3.xls, with monthly data on 3Month U.S. Tbill rates.

(a) Using data from 1960:1 – 1978:12. Using an ADF test constructed from the regression

$$R_t = \beta_0 + \lambda R_{t-1} + \sum_{i=1}^{4} \beta_i \Delta R_{t-i} + \varepsilon_t$$

test the null hypothesis that the process has a unit root.

(b) Repeat (a) using data from 1983:1 – 2004:12.

(c) Why was a constant term included in the regression?

(d) Using the ADF statistics from (a) and (b) compute 90% confidence intervals for the largest AR root over the two sample periods.

7. Using the data on U.S. GDP from 1959:1 – 2004:4, carry out the following calculations

(a) Compute growth rates $x_t = \ln(GDP_t / GDP_{t-1})$

(b) Estimate the AR(4) model $x_t = \beta + \sum_{i=1}^{4} \phi_i x_{t-i} + \varepsilon_t$

(c) Compute $y_t = \dfrac{x_t - \sum_{i=1}^{4} \hat{\phi}_i x_{t-i}}{\hat{\sigma}_\varepsilon}$

(d) Consider the model $y_t = \mu_t + e_t$, and carry out a Quandt test for a discrete break in μ using the middle 70% of the sample.

(e) Compute the p-value of the test-statistic in (d) using the file CDF_Quandt.xls.

(f) Suppose $\mu_t = \mu_{t-1} + (g/T)\eta_t$, where η_t is iid(0,1). Using the test statistic from (d) compute a median unbiased estimate of g. Construct a 90% confidence interval for g.

(g) Using the results in (f), how much time variation is present the "mean" of x_t?

(h) Considering you answer to (4), how should the procedure be modified to account for the time variation in the variance of ε_t?